XVII. *Memoir on the Theory of the Compositions of Numbers.*

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§ 1. *Unipartite Numbers.*

1. Compositions are merely partitions in which the order of occurrence of the parts is essential; thus, while the partitions of the number 3 are (3), (21), (111), the compositions are (3), (21), (12), (111).

The enumerations of the compositions of a number n into p parts, zeros excluded, is given by the coefficient of x^n in the expansion of

$$(x + x^2 + x^3 + \ldots)^p;$$

this expression may be written

$$\left(\frac{x}{1-x}\right)^p,$$

and the coefficient of x^n is seen to be

$$\binom{n-1}{p-1}.^{*}$$

The generating function of the total number of compositions of n is

$$\sum_1^\infty (x + x^2 + x^3 + \ldots)^p = \frac{x}{1-2x},$$

hence the number in question is

$$2^{n-1}.$$

2. If the parts of the compositions are limited not to exceed s in magnitude, the generating function of the number into p parts is

$$(x + x^2 + x^3 + \ldots + x^s)^p = x^p\left(\frac{1-x^s}{1-x}\right)^p,$$

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[*] In the continental notation I write $\binom{n}{p}$ for $\frac{n!}{p!(n-p)!}$.

and herein the coefficient of x^n is

$$\binom{n-1}{p-1} - \binom{p}{1}\binom{n-s-1}{n-s-p} + \binom{p}{2}\binom{n-2s-1}{n-2s-p} - \dots$$

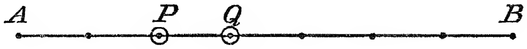
The number of parts being unrestricted the generating function is

$$\sum_p x^p \left(\frac{1-x^s}{1-x}\right)^p = \frac{x(1-x^s)}{1-2x+x^{s+1}}.$$

The expression $\binom{n-1}{p-1}$ is unchanged by the substitution of $n-p+1$ for p; hence the numbers of compositions of n into p parts and into $n-p+1$ parts are identical.

3. The graph of a number n is taken to be a straight line divided at $n-1$ points into n equal segments.

The graph of a composition of the number n is obtained by placing nodes at certain of these $n-1$ points of division.



AB being the graph of the number 7, for the representation of the composition (214), nodes are placed at the points P, Q, so that in moving from A to B by steps proceeding from node to node, 2, 1, and 4 segments of the line are passed over in succession. Although strictly speaking the initial and final points A, B are nodes on the graphs of all the compositions, it is only the inter-terminal nodes that will be considered in what follows, as appertaining to the graph.

The number of parts in the composition exceeds by unity the number of nodes on the graph.

For a composition of n into p parts we can place nodes at any $p-1$ out of the $n-1$ points of the graph of the number. The number of such composition graphs is at once seen to be

$$\binom{n-1}{p-1},$$

and further, since each of the $n-1$ points of the number graph is or is not the position of a node, the total number of composition graphs is

$$2^{n-1}.$$

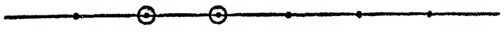
4. Associated with any one graph, there is another graph obtained by obliterating the nodes and placing nodes at the points not previously occupied.

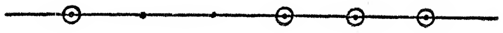
These graphs are said to be conjugate.

If a graph denotes a composition of n into p parts, the conjugate graph denotes a composition of n into $n-p+1$ parts.

This notion supplies, in consequence, a graphical proof of the theorem of Art. 2. Compositions of a number are conjugate when their graphs are conjugate.

E.g. The conjugate graphs





yield the conjugate compositions

(214) (13111)

The composition conjugate to a given composition may be written down, without constructing the graph, by the rules about to be explained.

The composition must first be prepared—

(i.) By writing successions of units in the power symbolism; for example, s successive units must be written 1^s;

(ii.) By intercalating 1^0 between each successive pair of non-unitary parts; thus, when aa or ab occur (a and b being superior to unity) we have to write $a1^0a$, $a1^0b$ respectively.

For the moment, call the non-unitary parts and the symbolic powers of unity the "elements" of the composition.

When an element does not occur at either end of the composition it is called "non-terminal," when at one end only "terminal," and when at both ends (thus constituting the entire composition), "doubly terminal."

The rules for procession to the conjugate are :—

I. If m or 1^m be doubly terminal, substitute 1^m for m or m for 1^m.

II. If m or 1^m be terminal, substitute 1^{m-1} for m or $m+1$ for 1^m.

III. If m or 1^m be non-terminal, substitute 1^{m-2} for m or $m+2$ for 1^m.

The composition thus obtained is in the "prepared" form and can be transformed to the ordinary form.

E.g. To find the conjugate of (231141), take the "prepared" form

$$(21^031^241),$$

and, beginning from the left, by

Rule	II.	For	2	substitute	1,
,,	III.	,,	1^0	,,	2,
,,	III.	,,	3	,,	1,
,,	III.	,,	1^2	,,	4,
,,	III.	,,	4	,,	1^2,
,,	II.	,,	1	,,	2,

resulting in the conjugate composition,

(12141^22), or non-symbolically (1214112)

An examination of the rules shows that they are reversible, and that the process gives a one-to-one correspondence between compositions of n into p and $n - p + 1$ parts.

A composition in general has, when prepared for conjugation, four different forms, viz. :—

$$(1)\ a_1 1^{\alpha_1} a_2 1^{\alpha_2} \ldots a_{s-1} 1^{\alpha_{s-1}} a_s.$$

$$(2)\ a_1 1^{\alpha_1} a_2 1^{\alpha_2} \ldots a_{s-1} 1^{\alpha_{s-1}} a_s 1^{\alpha_s}$$

$$(3)\ 1^{\alpha_1} a_2 1^{\alpha_2} \ldots a_{s-1} 1^{\alpha_{s-1}} a_s$$

$$(4)\ 1^{\alpha_1} a_2 1^{\alpha_2} \ldots a_{s-1} 1^{\alpha_{s-1}} a_s 1^{\alpha_s};$$

in all the four forms $a_1, a_2, a_3, \ldots a_s$ may have any positive integral values superior to unity. The numbers, $\alpha_1, \alpha_2, \ldots \alpha_s$, may have any positive integral values, including zero, with the exceptions,

In form (2) α_s cannot be zero.
(3) α_1 ,,
(4) α_1 and α_s ,,

The conjugates of the forms are

$$(1)\ 1^{a_1-1} . \alpha_1 + 2 . 1^{a_2-2} . \alpha_2 + 2 \ldots 1^{a_{s-1}-2} . \alpha_{s-1} + 2 . 1^{a_{s-1}-1},$$

$$(2)\ 1^{a_1-1} . \alpha_1 + 2 . 1^{a_2-2} . \alpha_2 + 2 \ldots 1^{a_{s-1}-2} . \alpha_{s-1} + 2 . 1^{a_s-2} . \alpha_s + 1,$$

$$(3)\ \alpha_1 + 1 . 1^{a_2-2} . \alpha_2 + 2 \ldots 1^{a_{s-1}-2} . \alpha_{s-1} + 2 . 1^{a_s-1},$$

$$(4)\ \alpha_1 + 1 . 1^{a_2-2} . \alpha_2 + 2 \ldots 1^{a_{s-1}-2} . \alpha_{s-1} + 2 . 1^{a_s-2} . \alpha_3 + 1.$$

5. Two compositions are said to be inverse (the one of the other) when the parts of the one, read from left to right, are identical with those of the other when read from right to left.

A composition may therefore be self-inverse.

In the graph of a self-inverse composition, the nodes must be symmetrically placed with respect to the extremities of the graph. If the number be even, the number of segments of the graph is even, and the two central nodes (nodes nearest to the centre of the graph) may be coincident, or they may include 2, 4, or any even number of segments. A self-inverse composition of an even number, say $2m$, into an even number, say $2p$, of parts, can only occur when the two central nodes of the graph are coincident and, attending to one side only of this node, we find that the number of self-inverse compositions of the number $2m$, composed of $2p$ parts, is equal to the number of compositions of m composed of p parts. In a notation, which is self-explanatory, we may write

$$\mathrm{SIC}\,(2m,\,2p) = \mathrm{C}\,(m,\,p) = \binom{m-1}{p-1}.$$

Next consider the self-inverse compositions of $2m$ into an uneven number, $2p - 1$, of parts. The two central nodes must be distinct, and may include any even number of segments. If this even number be 2κ the corresponding number of self-inverse compositions is equal to the number of compositions of $m - \kappa$ into $p - 1$ parts.

Hence

$$\mathrm{SIC}\,(2m,\,2p-1) = \mathrm{C}\,(m-1,\,p-1) + \mathrm{C}\,(m-2,\,p-2) + \ldots + \mathrm{C}\,(p-1,\,p-1),$$

or

$$\mathrm{SIC}\,(2m,\,2p-1) = \mathrm{C}\,(m,\,p) = \binom{m-1}{p-1}.$$

Self-inverse compositions of uneven numbers occur only when the number of parts is uneven, and it is easy to prove that

$$\mathrm{SIC}\,(2m-1,\,2p-1) = \mathrm{C}\,(m,\,p) = \binom{m-1}{p-1}.$$

Hence, without restriction of the number of parts,

$$\mathrm{SIC}\,(2m) = \mathrm{SIC}\,(2m+1) = \mathrm{C}\,(m+1) = 2^{m}.$$

This completes the enumeration of the self-inverse compositions.

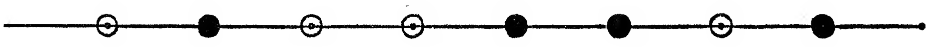
6. Two compositions which are at once conjugate and inverse, may be termed "inverse conjugates."

A composition whose conjugate is its own inverse is said to be "inversely conjugate."

Inversely conjugate compositions of a number n which have p parts, can occur only when $p = n - p + 1$, or $n = 2p - 1$, an uneven number.

The inversely conjugate compositions of $2m + 1$ are composed of $m + 1$ parts.

Consider a graph in which white and black nodes have reference to the two inverse conjugates respectively.



The black nodes are placed to the right and left of the centre of the graph in a manner similar to the white nodes to the left and right. If on the right there are

s black nodes and $m - s$ white nodes; to the left there are similarly placed s white nodes and $m - s$ black nodes.

Of the graph of the number there are m points to the right of the centre at which white and black nodes can be placed in 2^m distinct ways. Hence it at once follows that the number $2m + 1$ possesses in all 2^m inversely conjugate compositions. Otherwise, we may say that the number of inversely conjugate compositions of $2m + 1$ is equal to the number of compositions of $m + 1$.

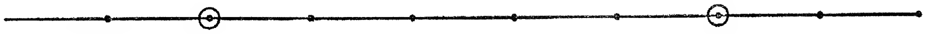
It will be observed that the number $2m + 1$ has precisely the same number, 2^m, of self-inverse compositions.

There is, in fact, a one-to-one correspondence between the compositions of $2m + 1$, which are inversely conjugate, and those which are self-inverse.

To explain this, take the graph last represented. Read according to the black nodes we obtain the inversely conjugate composition

(23121) of the number 9.

To proceed to the corresponding self-inverse composition obliterate the black nodes to the right of the centre, and also the white nodes to the left of the centre Substituting white nodes for the black nodes then remaining we have the graph



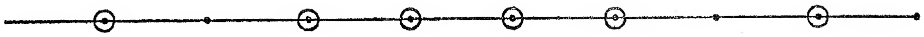
of the self-inverse composition

(252).

Again, reading the original graph according to white nodes, we have the inversely-conjugate composition

(12132),

and proceeding, as before, with the exception that black and white nodes are obliterated on the left and right of the centre respectively, we obtain the graph



of the self-inverse composition

(1211121).

The process is a general one, and shows that we can always pass from a composition which is inversely conjugate to one which is self-inverse.

E.g., of the number 7 we have the correspondence

Inversely conjugate,	Self inverse.
(4111)	(7)
(3211)	(313)
(2221)	(232)
(2131)	(21112)
(1312)	(151)
(1222)	(12121)
(1123)	(11311)
(1114)	(1111111)

The general form of an inversely conjugate composition is

$$\alpha_1 1^{\alpha_1} \alpha_2 1^{\alpha_2} \alpha_3 1^{\alpha_3} \ldots \alpha_3 + 2 \,.\, 1^{\alpha_3 - 2} \,.\, \alpha_2 + 2 \,.\, 1^{\alpha_2 - 2} \,.\, \alpha_1 + 2 \,.\, 1^{\alpha_1 - 1}$$

in its form prepared for conjugation.

7. The compositions of a number, m, give rise to the compositions of $m + 1$ by rules somewhat similar to those in ARBOGAST's method of derivations. The rules are obvious as soon as stated.

Each composition of the number m gives rise to two compositions of the number $m + 1$.

I. By prefixing the part unity.

II. By increasing the magnitude of the first part by unity.

All the compositions thus obtained are necessarily distinct. As an example see the subjoined scheme for passing from the compositions of 3 to those of 4.

111 12 21 3.

1111 211 112 22 121 31 13 4.

If the conjugates of these two lines be taken, the result is the same as the two lines inverted. We have the theorem, easily proved: "The conjugate of the $\genfrac{}{}{0pt}{}{\text{first}}{\text{second}}$ derivative of a composition is the $\genfrac{}{}{0pt}{}{\text{second}}{\text{first}}$ derivative of the conjugate composition."

8. The theory of the compositions of numbers is closely connected with the theory of the *perfect partitions* of numbers.* The connection is between the compositions of all multipartite numbers and the perfect partitions of unipartite numbers. The enumeration of the compositions of a single multipartite number enumerates also the perfect partitions of an infinite number of unipartite numbers. There is, moreover,

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* "The Theory of Perfect Partitions of Numbers and the Compositions of Multipartite Numbers," The Author, 'Messenger of Mathematics.' New Series, No. 235. November, 1890.

as was shown, *loc. cit.*, a one-to-one correspondence between the compositions of the unipartite number m and the perfect partitions, comprising m parts, of the whole assemblage of unipartite numbers.

Defining a perfect partition of a number to be one which contains one, and only one, partition of every lower number, it was shown that if

$$(\alpha, \beta, \gamma, \delta, \ldots)$$

be any composition of the number

$$\alpha + \beta + \gamma + \delta + \ldots,$$

the partition,

$$\{1^\alpha \,.\, (1+\alpha)^\beta \,.\, (1+\alpha \,.\, 1+\beta)^\gamma \,.\, (1+\alpha \,.\, 1+\beta \,.\, 1+\gamma)^\delta \ldots\},$$

the exponents being symbolic, denoting repetitions of parts, is a perfect partition of the number

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)\ldots - 1.$$

§ 2. *Multipartite Numbers.*

9. The multipartite number $\overline{\alpha\beta\gamma\ldots}$ may be regarded as specifying $\alpha + \beta + \gamma + \ldots$ things, α of one sort, β of a second, γ of a third, and so forth.

To illustrate partitions and compositions of such numbers, I write down those appertaining to the bipartite number $\overline{21}$.

Partitions.	Compositions.
$(\overline{21})$	$(\overline{21})$
$(\overline{20}\,\overline{01})$	$(\overline{20}\,\overline{01})$, $(\overline{01}\,\overline{20})$
$(\overline{11}\,\overline{10})$	$(\overline{11}\,\overline{10})$, $(\overline{10}\,\overline{11})$
$(\overline{10}^2\,\overline{01})$	$(\overline{10}^2\,\overline{01})$, $(\overline{10}\,\overline{01}\,\overline{10})$, $(\overline{01}\,\overline{10}^2)$

I speak of the parts of the partition or composition, and observe that each part is a multipartite number of the same nature, or order of multiplicity, as the number partitioned.

There is (*see* Art. 8, *loc. cit.*) a one-to-one correspondence between the compositions of the multipartite

$$\overline{\alpha\beta\gamma\ldots}$$

and the perfect partitions of the unipartite number

$$a^\alpha b^\beta c^\gamma \ldots - 1,$$

where $a, b, c, \ldots$ are any different prime numbers.

This correspondence is entirely distinct from that alluded to in Art. 12.

10. Taking for the present the general multipartite number to be

$$\overline{p_1 p_2 p_3 \ldots}$$

it is easily seen that the enumeration of the compositions into r parts is the same problem as the enumeration of the distributions of $p_1 + p_2 + p_3 + \ldots$ things, of which p_1 are of one sort, p_2 of a second, p_3 of a third, and so forth into r different parcels.

This number* is the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3} \ldots$ in the expansion of

$$(h_1 + h_2 + h_3 + \ldots)^r$$

wherein h_s denotes the sum of the homogeneous products, of degree s, of the quantities

$$\alpha_1, \alpha_2, \alpha_3 \ldots$$

Hence, the generating function of the total number of compositions is

$$\frac{h_1 + h_2 + h_3 + \ldots}{1 - h_1 - h_2 - h_3 - \ldots}.$$

The coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3} \ldots$ in the expansion of $(h_1 + h_2 + h_3 + \ldots)^r$ is readily found to be

$$\begin{aligned} &\binom{p_1 + r - 1}{p_1}\binom{p_2 + r - 1}{p_2}\binom{p_3 + r - 1}{p_3}\ldots \\ -&\binom{r}{1}\binom{p_1 + r - 2}{p_1}\binom{p_2 + r - 2}{p_2}\binom{p_3 + r - 2}{p_3}\ldots \\ +&\binom{r}{2}\binom{p_1 + r - 3}{p_1}\binom{p_2 + r - 3}{p_2}\binom{p_3 + r - 3}{p_3}\ldots \\ -&\ldots \text{ to } r \text{ terms.} \end{aligned}$$

and the enumeration is analytically complete.

11. This method is laborious in the case of high multipartite numbers since r may have all values from unity to $p_1 + p_2 + p_3 + \ldots$; but fortunately the series, above written, possesses some remarkable properties which can be utilized so as greatly to abridge the necessary labour.

Let $F(p_1 p_2 p_3 \ldots)$ and $f(p_1 p_2 p_3 \ldots, r)$ denote, respectively, the total number of compositions and the number comprised of r parts of the multipartite number $\overline{p_1p_2p_3 \ldots}$; so that,

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* See the Author, "Symmetric Functions and the Theory of Distributions," 'Proceedings of the London Mathematical Society,' vol. 19, Nos. 318–320.

$$F(p_1p_2p_3 \ldots) = \sum_1^{\Sigma p} f(p_1p_2p_0 \ldots, r).$$

Writing

$$(1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3) \ldots = 1 - a_1 + a_2 - a_3 + \ldots$$

$$\frac{h_1 + h_2 + h_3 + \ldots}{1 - h_1 - h_2 - h_3 - \ldots} = \frac{a_1 - a_2 + a_3 - \ldots}{1 - 2(a_1 - a_2 + a_3 - \ldots)}.$$

This new form of the generating function gives a relation connecting the number of compositions of any multipartite with numbers related to lower multipartites.

For unipartite numbers

$$\frac{\alpha_1}{1 - 2\alpha_1} = \Sigma F(p_1) \alpha_1^{p_1}$$

yielding

$$F(1) = 1$$

and

$$F(p_1) = 2F(p_1 - 1) \text{ when } p_1 > 1.$$

For bipartite numbers

$$\frac{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}{1 - 2(\alpha_1 + \alpha_2 - \alpha_1\alpha_2)} = \Sigma F(p_1p_2) \alpha_1^{p_1}\alpha_2^{p_2}$$

giving

$$F(p_1p_2) = 2F(p_1 - 1, p_2) + 2F(p_1, p_2 - 1) - 2F(p_1 - 1, p_2 - 1),$$

and similarly for multipartite numbers

$$\begin{aligned} F(p_1p_2p_3 \ldots) = & \; 2\{F(p_1 - 1, p_2, p_3, \ldots) + \ldots\} \\ & - 2\{F(p_1 - 1, p_2 - 1, p_3, \ldots) + \ldots\} \\ & + 2\{F(p_1 - 1, p_2 - 1\ p_3 - 1, \ldots) + \ldots\} \\ & - \ldots \end{aligned}$$

a formula absolutely true for all multipartites superior to $\overline{111 \ldots}$, and universally true if $F(000 \ldots)$ when it occurs be interpreted to mean $\frac{1}{2}$.*

12. A simple expression is obtainable for $F(p_1p_2)$. We have to find the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}$ in

$$\frac{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}{1 - 2(\alpha_1 + \alpha_2 - \alpha_1\alpha_2)}$$

or, this is the coefficient of $\alpha_1^{p_1}$ in

$$2^{p_2 - 1} \frac{(1 - \alpha_1)}{(1 - 2\alpha_1)^{p_2 + 1}},$$

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* See *post*, Art. 39.

and thence

$$\mathrm{F}(p_1p_2) = 2^{p_1+p_2-1}\frac{(p_1+p_2)!}{p_1!\,p_2!} - 2^{p_1+p_2-2}\frac{(p_1+p_2-1)!}{1!\,(p_1-1)!\,(p_2-1)!}$$

$$+ 2^{p_1+p_2-3}\frac{(p_1+p_2-2)!}{2!\,(p_1-2)!\,(p_2-2)!} - \ldots$$

until one of the denominator factorials becomes zero.*

13. There is another method by which the value of $\mathrm{F}(p_1p_2p_3\ldots\mu)$ can be easily obtained from the numbers $f(p_1p_2p_3\ldots, r)$ which compose the value of $\mathrm{F}(p_1p_2p_3\ldots)$. Writing $h_1 + h_2 + h_3 + \ldots = \mathrm{H}$, we may write the generating function

$$\frac{\mathrm{H}}{1-\mathrm{H}} = \Sigma\,\mathrm{F}(p_1p_2p_3\ldots).(p_1p_2p_3\ldots)$$

where $(p_1p_2p_3\ldots)$ denotes the symmetric function

$$\Sigma\,\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$$

and

$$\mathrm{H}^r = \Sigma f(p_1p_2p_3\ldots, r).(p_1p_2p_3\ldots).$$

Let

$$d_r = \partial_{\alpha_r} + \alpha_1\partial_{\alpha_{r+1}} + \alpha_2\partial_{\alpha_{r+2}} + \ldots$$

$$\mathrm{D}_r = \frac{1}{r!}(\partial_{\alpha_1} + \alpha_1\partial_{\alpha_2} + \alpha_2\partial_{\alpha_3} + \ldots)^r,$$

D_r being an operator of the r^{th} order obtained by symbolical multinomial expansion, as in TAYLOR'S theorem of the Differential Calculus, and not denoting r successive performances of linear operations. The effect of the operation of D_r upon a symmetric function of the quantities $\alpha_1, \alpha_2, \alpha_3 \ldots$ expressed in the notation of partitions, is well known; it obliterates one part r from every symmetric function partition which possesses such a part, and causes every other symmetric function to vanish.

Also $\mathrm{D}_r(r) = 1$.

Hence

$$\mathrm{D}_\mu\mathrm{H}^r = \overset{p}{\Sigma} f(p_1p_2p_3\ldots\mu, r).(p_1p_2p_3\ldots).$$

To evaluate $\mathrm{D}_\mu\mathrm{H}^r$ we require the well-known formulæ

$$d_\mu h_s = (-)^{\mu+1}h_{s-\mu},$$

leading to

$$d_\mu\mathrm{H} = (-)^{\mu+1}(1+\mathrm{H}).$$

When operating upon a function of H, d_μ is equivalent to d_1, when μ is uneven, and equivalent to $-d_1$ when μ is even. Hence, with such an operand, we have the equivalences

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* Tables for the verification of formulæ will be found *post* pp. 898–900.

$$D_1 \equiv d_1,$$

$$D_2 \equiv \frac{1}{2!} d_1 (d_1 + 1),$$

$$D_3 \equiv \frac{1}{3!} d_1 (d_1 + 1)(d_1 + 2),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$D_\mu \equiv \frac{1}{\mu!} d_1 (d_1 + 1) \ldots (d_1 + \mu - 1),$$

the products on the right denoting successive operations.*

To evaluate $D_\mu H^r$ as an algebraic function of H, it is necessary to operate successively with d_1, $d_1 + 1, \ldots d_1 + \mu - 1$ and to divide the result by $\mu!$

We have

$$D_1 H^r = r H^{r-1} + r H^r,$$

$$D_2 H^r = \binom{r}{2} H^{r-2} + \binom{r}{1}^2 H^{r-1} + \binom{r+1}{2} H^r.$$

Suppose that generally

$$D_\mu H^r = \sum_{s=0}^{s=\mu} \binom{r+s-1}{s} \binom{r}{\mu - s} H^{r-\mu+s}.$$

Operation with

$$\frac{1}{\mu + 1}(d_1 + \mu)$$

gives

$$D_{\mu+1} H^r = \sum_{s=0}^{s=\mu+1} \binom{r+s-1}{s} \binom{r}{\mu + 1 - s} H^{r-(\mu+1)+s}.$$

so that the assumed law is established inductively.

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* The truth of this law is seen by comparison with the corresponding algebraic relations. Denoting by $s_1, s_2, \ldots$ the sums of powers of $\alpha_1, \alpha_2, \alpha_3, \ldots$

$$1 - a_1 x + a_2 x^2 - \ldots = \exp. - (s_1 x + \tfrac{1}{2} s_2 x^2 + \tfrac{1}{3} s_3 x^3 + \ldots),$$

and putting

$$s_{2\kappa+1} = s_1, \quad s_{2\kappa} = - s_1,$$

$$1 - a_1 x + a_2 x^2 - \ldots = (1 + x)^{-s_1}.$$

Hence

$$a_\mu = \frac{1}{\mu!} s_1 (s_1 + 1) \ldots (s_1 + \mu - 1),$$

from which we pass to the operator relation

$$D_\mu = \frac{1}{\mu!} d_1 (d_1 + 1) \ldots (d_1 + \mu - 1).$$

Observe also the relation

$$h_\mu = \frac{1}{\mu!} s_1 (s_1 - 1) \ldots (s_1 - \mu + 1).$$

Substituting in a previous identity

$$\sum_{s=0}^{s=\mu} \binom{r+s-1}{s} \binom{r}{\mu-s} \mathrm{H}^{r-\mu+s} = \sum^{p} f(p_1 p_2 p_3 \ldots \mu, r) \,.\, (p_1 p_2 p_3 \ldots),$$

therefore,

$$\sum_{s=0}^{s=\mu} \binom{r+s-1}{s} \binom{r}{\mu-s} \sum^{p} f(p_1 p_2 p_3 \ldots, r-\mu+s) \,.\, (p_1 p_2 p_3 \ldots)$$
$$= \sum^{p} f(p_1 p_2 p_3 \ldots \mu, r) \,.\, (p_1 p_2 p_3 \ldots).$$

This is an absolute identity and, equating coefficients after writing

$$\binom{r+s-1}{s} \binom{r}{\mu-s} = \phi(r, s),$$

we obtain

$$\phi(r,\mu) f(p_1 p_2 p_3 \ldots, r) + \phi(r, \mu-1) f(p_1 p_2 p_3 \ldots, r-1) + \ldots$$
$$+ \phi(r, \mu-r+1) f(p_1 p_2 p_3 \ldots, 1) = f(p_1 p_2 p_3 \ldots \mu, r).$$

This formula enables the calculation of the number $f(p_1 p_2 p_3 \ldots \mu, r)$ from the successive numbers

$$f(p_1 p_2 p_3 \ldots, r), \quad f(p_1 p_2 p_3 \ldots, r-1), \ldots f(p_1 p_2 p_3 \ldots, 1).$$

14. A more useful result is obtained by summing each side of this identity for a values of r from 1 to $\Sigma p + \mu$. This result is

$$\sum^{r} \{\phi(r,\mu) + \phi(r+1, \mu-1) + \ldots + \phi(r+\mu, 0)\} f(p_1 p_2 p_3 \ldots r) = \mathrm{F}(p_1 p_2 p_3 \ldots \mu).$$

It can be shown that the expression in brackets { } to the left has the values

$$2^{\mu-1} \binom{r+\mu-1}{r} \frac{2r+\mu}{\mu};$$

therefore,

$$\mathrm{F}(p_1 p_2 p_3 \ldots \mu) = \sum^{r} 2^{\mu-1} \binom{r+\mu-1}{r} \frac{2r+\mu}{\mu} f(p_1 p_2 p_3 \ldots, r)$$
$$= 2^{\mu-1}(\mu+2) f(p_1 p_2 p_3 \ldots, 1) + 2^{\mu-1} \frac{(\mu+1)(\mu+4)}{2!} f(p_1 p_2 p_3 \ldots, 2)$$
$$+ 2^{\mu-1} \frac{(\mu+1)(\mu+2)(\mu+6)}{3!} f(p_1 p_2 p_3 \ldots, 3) + \ldots,$$

a formula of great service when μ is large, as the number of arithmetical operations is comparatively small.

As an example we can find another series for $\mathrm{F}(p_1 p_2)$.

Since

$$\mathrm{F}(p_1) = 1 + \binom{p_1 - 1}{1} + \binom{p_1 - 1}{2} + \ldots + 1$$

wherein

$$\binom{p_1 - 1}{r} \text{ is the value of } f(p_1, r),$$

we find

$$\mathrm{F}(p_1 p_2) = 2^{p_2 - 1}(p_2 + 2) + 2^{p_2 - 1}\frac{(p_2 + 1)(p_2 + 4)}{2!}(p_1 - 1)$$
$$+ 2^{p_2 - 1}\frac{(p_2 + 1)(p_2 + 2)(p_2 + 6)}{3!}\binom{p_1 - 1}{2} + \ldots,$$

a series which it is easy to identify with that previously given (Art. 12).

15. Useful formulæ of verification are obtainable.

It has been shown in Art. 13 that the operation $(-)^{\mu+1} d_\mu$ is equivalent to d_1 when the operand is a function of H only.

Now

$$\frac{(-)^{\mu+1}}{\mu} d_\mu = \Sigma (-)^{\alpha+\beta+\ldots-1}\frac{(\alpha + \beta + \ldots - 1)!}{\alpha!\,\beta!\ldots}\mathrm{D}_1^{\alpha}\mathrm{D}_2^{\beta}\ldots,$$

the summation having reference to all positive integer solutions of the equation,

$$\alpha + 2\beta + 3\gamma + \ldots = \mu.$$

Operating on the expression

$$\Sigma f(p_1 p_2 p_3 \ldots, r) . (p_1 p_2 p_3 \ldots)$$

with $(-)^{\mu+1} d_\mu$ for successive positive integral values of μ, and equating the coefficients of the symmetric function $(p_1 p_2 p_3 \ldots)$ we find the relations—

$$\begin{aligned} & f(p_1 p_2 p_3 \ldots 1, r) \\ & \quad = -\{f(p_1 p_2 p_3 \ldots 1^2, r) - 2f(p_1 p_2 p_3 \ldots 2, r)\} \\ & \quad = +\{f(p_1 p_2 p_3 \ldots 1^3, r) - 3f(p_1 p_2 p_3 \ldots 21, r) + 3f(p_1 p_2 p_3 \ldots 3, r)\} \\ & \quad = - \ldots \\ & \quad = (-)^{s+1}\Sigma(-)^{\alpha+\beta+\ldots-1}\frac{(\alpha + \beta + \ldots - 1)!\, s}{\alpha!\,\beta!\ldots} f(p_1 p_2 p_3 \ldots 2^{\beta} 1^{\alpha}, r), \end{aligned}$$

the condition of summation being

$$\alpha + 2\beta + 3\gamma + \ldots = s.$$

Either summing these identities with respect to r, or operating as before upon the expression

$$\Sigma\, \mathrm{F}(p_1p_2p_3\ldots).(p_1p_2p_3\ldots),$$

we obtain the identities

$$\begin{aligned}
&\mathrm{F}(p_1p_2p_3\ldots 1)\\
&\quad = -\{\mathrm{F}(p_1p_2p_3\ldots 1^2) - 2\mathrm{F}(p_1p_2p_3\ldots 2)\}\\
&\quad = +\{\mathrm{F}(p_1p_2p_3\ldots 1^3) - 3\mathrm{F}(p_1p_2p_3\ldots 21) + 3\mathrm{F}(p_1p_2p_3\ldots 3)\}\\
&\quad = \ldots\\
&\quad = (-)^{s+1}\,\Sigma\,(-)^{\alpha+\beta+\ldots-1}\frac{(\alpha+\beta+\ldots-1)!\,s}{\alpha!\,\beta!\ldots}\mathrm{F}(p_1p_2p_3\ldots 2^\beta 1^\alpha).
\end{aligned}$$

These relations are readily verified in the particular case

$$(p_1p_2p_3\ldots 1) = (1).$$

16. Another very useful result is derived from the algebraical result noticed in the foot-note to Art. 13.

Since the supposition $(-)^{\mu-1}s_\mu = s_1$ leads to the formula

$$\frac{1}{\mu!}s_1(s_1-1)\ldots(s_1-\mu+1) = h_\mu$$

and

$$h_\mu = \Sigma\,(-)^{\mu+v_1+v_2+\ldots}\frac{(v_1+v_2+\ldots)!}{v_1!\,v_2!\ldots}a_1^{v_1}a_2^{v_2}\ldots$$

we reach the operator relation

$$\frac{1}{\mu!}d_1(d_1-1)\ldots(d_1-\mu+1) = \Sigma\,(-)^{\mu+v_1+v_2+\ldots}\frac{(v_1+v_2+\ldots)!}{v_1!\,v_2!\ldots}\mathrm{D}_1^{v_1}\mathrm{D}_2^{v_2}\ldots$$

whenever, as in the present case, the operand is such that $(-)^{\mu-1}d_\mu = d_1$.

Assuming

$$\frac{1}{\mu!}d_1(d_1-1)\ldots(d_1-\mu+1)\mathrm{H}^r = \binom{r}{\mu}\mathrm{H}^{r-\mu}(1+\mathrm{H})^\mu$$

and operating with $\frac{1}{\mu+1}(d_1-\mu)$ we find

$$\frac{1}{(\mu+1)!}d_1(d_1-1)\ldots(d_1-\mu)\mathrm{H}^r = \binom{r}{\mu+1}\mathrm{H}^{r-\mu-1}(1+\mathrm{H})^{\mu+1}$$

verifying the assumption.

Hence operation on the relation

$$\mathrm{H}^r = \Sigma f(p_1 p_2 p_3 \ldots, r) \cdot (p_1 p_2 p_3 \ldots)$$

yields the relation

$$\binom{r}{\mu} \mathrm{H}^{r-\mu} (1 + \mathrm{H})^{\mu}$$
$$= \Sigma (-)^{\mu + v_1 + v_2 + \ldots} \frac{(v_1 + v_2 + \ldots)!}{v_1!\, v_2! \ldots} \mathrm{D}_1^{v_1} \mathrm{D}_2^{v_2} \ldots \{\Sigma f(p_1 p_2 p_3 \ldots, r) \cdot (p_1 p_2 p_3 \ldots)\};$$

or

$$\binom{r}{\mu} \Sigma \left\{ f(p_1 p_2 p_3 \ldots, r - \mu) + \binom{\mu}{1} f(p_1 p_2 p_3 \ldots, r - \mu + 1) \right.$$
$$\left. + \binom{\mu}{2} f(p_1 p_2 p_3 \ldots, r - \mu + 2) + \ldots \right\} \cdot (p_1 p_2 p_3 \ldots)$$
$$= \Sigma \left\{ f(p_1 p_2 p_3 \ldots 1^{\mu}, r) - (\mu - 1) f(p_1 p_2 p_3 \ldots 21^{\mu-2}, r) \right.$$
$$+ \binom{\mu - 2}{1} f(p_1 p_2 p_3 \ldots 31^{\mu-3}, r) + \binom{\mu - 2}{2} f(p_1 p_2 p_3 \ldots 2^2 1^{\mu-4}, r)$$
$$\left. + \ldots \right\} \cdot (p_1 p_2 p_3 \ldots),$$

or, equating coefficients and inverting

$$f(p_1 p_2 p_3 \ldots 1^{\mu}, r) - (\mu - 1) f(p_1 p_2 p_3 \ldots 21^{\mu-2}, r)$$
$$+ (\mu - 2) f(p_1 p_2 p_3 \ldots 31^{\mu-3}, r) + \ldots$$
$$= \binom{r}{\mu} \left\{ f(p_1 p_2 p_3 \ldots, r - \mu) + \binom{\mu}{1} f(p_1 p_2 p_3 \ldots, r - \mu + 1) \right.$$
$$\left. + \binom{\mu}{2} f(p_1 p_2 p_3 \ldots, r - \mu + 2) + \ldots \right\}.$$

If $r < \mu$ the dexter vanishes and

$$f(p_1 p_2 p_3 \ldots 1^{\mu}, r) - (\mu - 1) f(p_1 p_2 p_3 \ldots 21^{\mu-2}, r)$$
$$+ (\mu - 2) f(p_1 p_2 p_3 \ldots 31^{\mu-3}, r) + \ldots = 0.$$

If $r = \mu$ the dexter is

$$f(p_1 p_2 p_3 \ldots, 0) + \binom{\mu}{1} f(p_1 p_2 p_3 \ldots, 1) + \binom{\mu}{2} f(p_1 p_2 p_3 \ldots, 2) + \ldots$$

where $f(p_1 p_2 p_3 \ldots, 0)$ in general vanishes, but

$$f(0, 0) = 1.$$

Performing the summation in regard to r,

$$\mathrm{F}(p_1p_2p_3\ldots 1^{\mu}) - (\mu - 1)\,\mathrm{F}(p_1p_2p_3\ldots 21^{\mu-2}) + \ldots$$

$$+ (-)^{\mu+v_1+v_2+\ldots}\frac{(v_1+v_2+\ldots)!}{v_1!\,v_2!\ldots}\,\mathrm{F}(p_1p_2p_3\ldots 2^{v_2}1^{v_1}) + \ldots$$

$$= \binom{\mu}{0} f(p_1p_2p_3\ldots, 0) + \left\{\binom{\mu}{1} + \binom{\mu+1}{1}\binom{\mu}{0}\right\} f(p_1p_2p_3\ldots, 1) + \ldots$$

$$+ \theta_{\mu,s} f(p_1p_2p_3\ldots, s) + \ldots,$$

where

$$\theta_{\mu,s} = \binom{\mu}{s} + \binom{\mu+1}{1}\binom{\mu}{s-1} + \binom{\mu+2}{2}\binom{\mu}{s-2} + \ldots + \binom{\mu+s}{s}$$

that is to say

$$(1+x)^{\mu}\,(1-x)^{-\mu-1} = \Sigma\,\theta_{\mu,s}\,x^s.$$

E.g., $r < \mu$,

$$f(21^3, 2) - 2f(2^21, 2) + f(32, 2) = 0,$$

verified by

$$22 - 2 \times 16 + 10 = 0.$$ (See Tables, pp. 898–900.)

$r = \mu$,

$$f(21^3, 3) - 2f(2^21, 3) + f(32, 3) = f(2, 0) + 3f(2, 1) + 3f(2, 2)$$

verified by

$$93 - 2 \times 57 + 27 = 0 + 3 \times 1 + 3 \times 1.$$

For the summation formula

$$\begin{aligned}
&\mathrm{F}(1),\\
=\;&\mathrm{F}(1^2) - \mathrm{F}(2),\\
=\;&\mathrm{F}(1^3) - 2\mathrm{F}(21) + \mathrm{F}(3),\\
=\;&\mathrm{F}(1^4) - 3\mathrm{F}(21^2) + \mathrm{F}(2^2) + 2\mathrm{F}(31) - \mathrm{F}(4),\\
=\;&\ldots\ldots\\
=\;&1,
\end{aligned}$$

verified by

$$\begin{aligned}
&1,\\
=\;&3 - 2,\\
=\;&13 - 2 \times 8 + 4,\\
=\;&75 - 3 \times 44 + 26 + 2 \times 20 - 8,\\
=\;&\ldots\ldots\\
=\;&1;
\end{aligned}$$

also

$$\begin{aligned} &F(41^3) - 2F(421) + F(43) \\ &\quad = f(4, 0) + (3 + 4 \times 1) f(4, 1) + (3 + 4 \times 3 + 10 \times 1) f(4, 2) \\ &\qquad + (1 + 4 \times 3 + 10 \times 3 + 20 \times 1) f(4, 3) \\ &\qquad + (4 \times 1 + 10 \times 3 + 20 \times 3 + 35 \times 1) f(4, 4), \end{aligned}$$

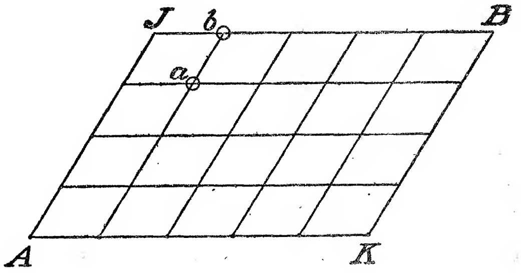
verified by

$$\begin{aligned} &3408 - 3776 + 768 \\ &= 0 + 7 \times 1 + 25 \times 3 + 63 \times 3 + 129 \times 1, \\ &= 400. \end{aligned}$$

§ 3. *The Graphical Representation of the Compositions of Bipartite Numbers.*

17. The graphical method, that has been employed in the case of unipartite compositions, can be extended so as to meet the cases of bipartite, tripartite, and multipartite numbers in general. For the present the bipartite case alone is under consideration. The graph of a bipartite number $(\overline{pq})$ is derived directly from the graphs of the unipartite numbers (p), (q).

Take $q + 1$ exactly similar graphs of the number p and place them parallel to one another, at equal distances apart, and so that their left hand extremities lie on a right line; corresponding points of the $q + 1$ graphs can then be joined by right lines and a reticulation will be formed which is the graph of the bipartite number $\overline{pq}$.



We have AK a graph of the number p, and $q + 1$ such graphs parallel to one another; and AJ a graph of the number q, and $p + 1$ such graphs parallel to one another.

The angle between AK and AJ is immaterial.

The points A, B, are the "extremities" or the "initial" and "final" points of the graph.

The remaining intersections are termed the "points" of the graph.

The lines of the graph have either the "direction" AK or the direction AJ. These will be called the α and β directions respectively. Through each point of the graph pass lines in each of these directions. Each line is made up of segments, and we speak of α segments and of β segments indicating that the corresponding lines (on which lie the segments) are in the α and β directions.

Suppose a traveller to proceed from A to B by successive steps. A step is performed by moving over a certain number of α segments and subsequently moving over a certain number of β segments. A step is thus made up of two figures—say an α figure and a β figure. The number of segments moved over may be zero in either, but not in both of these two figures of the step.

A step may be taken from A to any point a of the graph; a second step may be taken from a to any point of the graph aB which has a and B for its initial and final points; subsequent steps are taken on a similar principle, and the last step terminates at the point B and completes the procession from the point A to the point B. A step which takes x, α segments followed by y, β segments, is taken to be the representation of a bipartite part, $\overline{xy}$. A procession from the initial to the final point (from A to B) of the graph thus represents a sequence of bipartite parts which constitutes a composition of the bipartite number $\overline{pq}$. To every procession from A to B corresponds a composition of the bipartite $\overline{pq}$, and the enumeration of the different processions is identical with the enumeration of the total number of different compositions.

The steps of a procession are marked out by nodes placed at the points of the reticulation, which terminate the first, second, third, &c., and penultimate steps When nodes are thus placed, we have the graph of a composition. If nodes be placed at the points a, b of the graph, we obtain the graph of the composition

$$(\overline{13}\ \overline{01}\ \overline{40})$$

of the bipartite $\overline{54}$.

The number of parts in a composition is always one greater than the number of nodes in its graph.

Considering the number of compositions of $\overline{pq}$, of two parts, it is clear that we may place the defining node at any one of $(p+1)(q+1)-2$ points of the reticulation. Hence (and this may be verified by previous work) the number of two-part compositions is

$$(p+1)(q+1)-2.$$

18. The graph of a composition, traced from A to B, passes over certain segments, and may be said to follow a certain line of route through the reticulation. Other compositions in general follow the same line of route; they all have their defining nodes upon the line of route, and a certain number of defining nodes will, in general, be common to all of them.

Consider the path $AcbB$ in the graph given below. All compositions whose graphs follow this line of route must have the node b; b, in fact, is an essential node along this line of route.

Essential nodes occur on a line of route at all points where the course changes from the β to the α direction.

We may regard a line of route as defined by these essential nodes since the line of route is completely given by these nodes.

Every composition-graph involves nodes of two kinds—

(1) Those which are essential to its line of route;

(2) Those which, in this respect, are not essential.

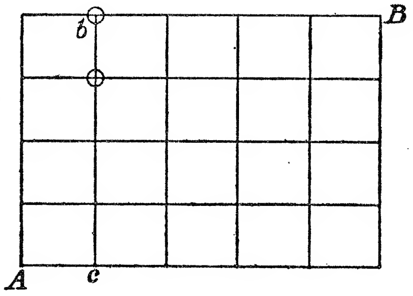
There are $p + q - 1$ "points" along every line of route.

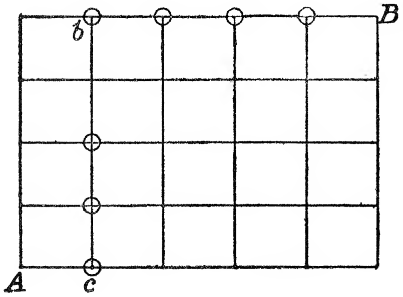
We have now to discuss the compositions whose graphs follow a given line of route, and we find that they naturally arrange themselves in pairs.

19. Associated with any one graph is another obtained by obliterating those nodes which are not essential to its line of route, and by then placing nodes at those points on the line of route not previously occupied by nodes.

These two graphs are said to be conjugate.

The compositions, represented by these graphs, are likewise said to be conjugate.





Conjugate graphs are shown above; $AcbB$ is the line of route, and b is the essential node.

The corresponding conjugate compositions of the bipartite number $\overline{54}$ are

$$(\overline{13}\ \overline{01}\ \overline{40}) \text{ and } (\overline{10}\ \overline{01}\ \overline{01}\ \overline{02}\ \overline{10}\ \overline{10}\ \overline{10}\ \overline{10}).$$

If the graph of a composition of $\overline{pq}$, of r parts, possesses s essential nodes, it is clear that the conjugate composition has $p + q - r + s + 1$ parts.

Of compositions, of the bipartite $\overline{pq}$, whose graphs possess s essential nodes, there is a one-to-one correspondence between those of r parts and those of $p + q - r + s + 1$ parts.

Corresponding to an essential node in the graph of a composition there exist in the composition itself adjacent parts . . . $\overline{p_1q_1}\ \overline{p_2q_2}$ . . . possessing the property of q_1 and p_2, being both superior to zero. Thus, from inspection of a composition, we are enabled to determine the number of essential nodes in its graph.

It is useful to recognise four species of contact between adjacent parts of a composition

$$\text{in} \ldots \overline{p_1q_1}\ \overline{p_2q_2} \ldots$$

if q_1	is zero and	p_2 zero	we have a	zero-zero	contact
q_1	,,	p_2 positive	,,	zero-positive	,,
q_1	positive	p_2 zero	,,	positive-zero	,,
q_1	,,	p_2 positive	,,	positive-positive	,,

In this nomenclature we may say that the graph of a composition possesses as many essential nodes as the composition itself possesses positive-positive contacts.

20. The theorem arrived at may be stated as follows :—

"Of compositions of $\overline{pq}$ possessing s positive-positive contacts, there is a one-to-one correspondence between those of r parts and those of $p + q - r + s + 1$ parts."

An essential node corresponding to a positive-positive contact in the composition occurs at a point of the graph where there is a change from a β direction to an α direction; we may say that this is a $\beta\alpha$ point on the line of route. Similarly to positive-zero, zero-positive, zero-zero contacts in the composition correspond $\beta\beta$, $\alpha\alpha$, $\alpha\beta$ points respectively on the line of route.

The number of different lines of route that can be traced on the graph of the bipartite number is the number of permutations of p symbols α, and q symbols β, for this is the number of ways in which the p α-segments and the q β-segments, which make up a line of route, can form a succession.

Hence the number of lines of route through the reticulation is

$$\binom{p+q}{p}.$$

The whole of the compositions of $\overline{pq}$ can be arranged in conjugate pairs.

E.g., The correspondence in regard to the compositions of the bipartite $\overline{22}$ is shown in parallel columns.

$$(\overline{22}) \quad s = 0, \quad r = 1 \qquad\qquad (\overline{10}\ \overline{10}\ \overline{01}\ \overline{01}) \quad s = 0, \quad r = 4$$

$$\left.\begin{matrix}(\overline{20}\ \overline{02})\\(\overline{21}\ \overline{01})\\(\overline{10}\ \overline{12})\end{matrix}\right\} \ s = 0, \quad r = 2 \qquad\qquad \left.\begin{matrix}(\overline{10}\ \overline{11}\ \overline{01})\\(\overline{10}\ \overline{10}\ \overline{02})\\(\overline{20}\ \overline{01}\ \overline{01})\end{matrix}\right\} \ s = 0, \quad r = 3$$

$$\left.\begin{array}{l}(\overline{02}\ \overline{20})\\(\overline{11}\ \overline{11})\\(\overline{01}\ \overline{21})\\(\overline{12}\ \overline{10})\end{array}\right\}\quad s=1,\quad r=2 \qquad \left.\begin{array}{l}(\overline{01}\ \overline{01}\ \overline{10}\ \overline{10})\\(\overline{10}\ \overline{01}\ \overline{10}\ \overline{01})\\(\overline{01}\ \overline{10}\ \overline{10}\ \overline{01})\\(\overline{10}\ \overline{01}\ \overline{01}\ \overline{10})\end{array}\right\}\quad s=1,\quad r=4$$

$$\left.\begin{array}{l}(\overline{01}\ \overline{20}\ \overline{01})\\(\overline{01}\ \overline{01}\ \overline{20})\\(\overline{10}\ \overline{02}\ \overline{10})\\(\overline{11}\ \overline{10}\ \overline{01})\end{array}\right\}\quad s=1,\quad r=3 \qquad \left.\begin{array}{l}(\overline{01}\ \overline{10}\ \overline{11})\\(\overline{02}\ \overline{10}\ \overline{10})\\(\overline{11}\ \overline{01}\ \overline{10})\\(\overline{10}\ \overline{01}\ \overline{11})\end{array}\right\}\quad s=1,\quad r=1$$

$$(\overline{01}\ \overline{11}\ \overline{10})\quad s=2,\quad r=3 \qquad (\overline{01}\ \overline{10}\ \overline{01}\ \overline{10})\quad s=2,\quad r=4.$$

The compositions present themselves in pairs of conjugate groups according to the several values of s and r. In the above example for $s=1$, $r=3$, there is a self-conjugate group. This happens when $2r=p+q+s+1$.

A self-conjugate group exists for even or uneven values of s, according as $p+q$ is uneven or even.

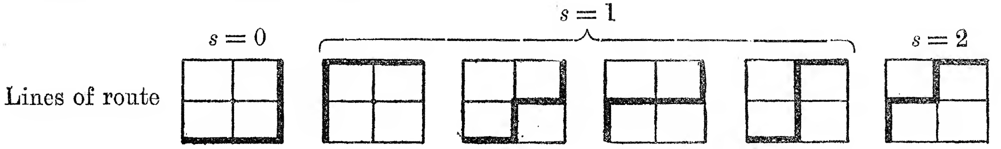
21. The enquiry now is in regard to the number of lines of route through the reticulation which possess exactly s essential nodes.

It may be observed, in passing, that from any line of route may be derived a composition whose graph exhibits only essential nodes and no others. This may be called the principal composition along the line of route; it will have $s+1$ parts, and each of the s contacts of its parts will be positive-positive.

There is a one-to-one correspondence between the lines of route having s essential nodes and the compositions of $s+1$ parts, all of whose contacts are positive-positive.

Also the number of compositions, number of parts unrestricted, all of whose contacts are positive-positive is equal to the number of different lines of route.

E.g., for the number $\overline{22}$,



Compositions $(\overline{22})$ $(\overline{02}\ \overline{20})$ $(\overline{11}\ \overline{11})$ $(\overline{01}\ \overline{21})$ $(\overline{12}\ \overline{10})$ $(\overline{01}\ \overline{11}\ \overline{10})$

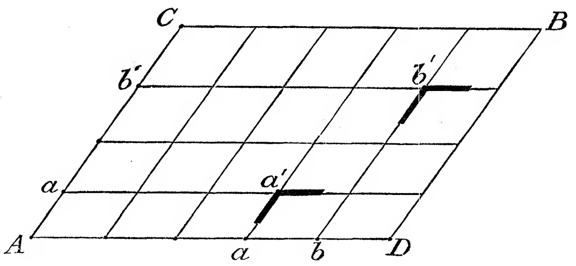
and there are no other compositions having all contacts positive-positive.

22. The number of different lines of route with exactly s essential nodes is

$$\binom{p}{s}\binom{q}{s}.$$

Of this theorem I give three proofs, because its thorough examination is necessary for the purpose of leading up to the more difficult theories connected with tripartite and higher multipartite numbers.

First Proof.



In each of the adjacent sides *AD*, *AC* of the graph of $\overline{pq}$, select any s "points" (see definition of "point") a, b, c, . . . in order from the point A. The two points a, a are seen to determine an essential node a'; the two points b, b an essential node b', &c., and a line of route necessarily exists which possesses these essential nodes and no others. The points along AD and AC, from which s points may be selected, are in number p and q respectively; along AD s points can be selected in $\binom{p}{s}$ ways, and along AC in $\binom{q}{s}$ ways; any selection on AD can be taken with any selection on AC. Hence the number of lines of route having exactly s essential nodes is

$$\binom{p}{s}\binom{q}{s}.$$

23. *Second Proof.*

We determined the total number of lines of route by considering the permutations of p, α-segments, and q, β-segments. Whenever in any such permutation there is a sequence $\beta\alpha$ there must be an essential node upon the corresponding line route. We have simply to find the number of permutations of the $p+q$ symbols in $\alpha^p\beta^q$ which possess exactly s, $\beta\alpha$-contacts.

Write down the s, $\beta\alpha$-sequences

$$\ldots\beta\alpha\ldots\beta\alpha\ldots\beta\alpha\ldots\beta\alpha\ldots$$

and the $s+1$ intervals between them. In these intervals we have to distribute the letters in

$$\alpha^{p-s}\ \beta^{q-s}$$

in such manner as to introduce no fresh $\beta\alpha$-contacts. For each of these $p+q-2s$ letters there is a choice of $s+1$ intervals. The $p-s$ letters α may thus be distributed in $\binom{p}{s}$ ways, and the $q-s$ letters β in $\binom{q}{s}$, and each distribution of the letters α may occur with each distribution of the letters β. Hence the total number of permutations is

$$\binom{p}{s}\ \binom{q}{s}.$$

24. *Third Proof.*[*]

I now come to a method which is valuable in the theory of multipartite numbers in general, and also intrinsically of great interest.

I consider, as in the second proof, the permutations of the $p + q$ letters $\alpha^p\beta^q$, which exhibit exactly s, $\beta\alpha$-contacts. The proof depends upon showing there exists a one-to-one correspondence between these permutations, and those in which the letter β occurs exactly s times in the first p places counted from the left.

Suppose the permutation with s, $\beta\alpha$-contacts to be

$$\alpha^{x_1}\beta^{y_1}\quad \beta\alpha\quad \alpha^{x_2}\beta^{y_2}\quad \beta\alpha\quad \alpha^{x_3}\beta^{y_3}\quad \beta\alpha\quad \alpha^{x_4}\beta^{y_4}\quad \beta\alpha\quad \alpha^{x_5}\beta^{y_5}\quad \beta\alpha\quad \alpha^{x_6}\beta^{y_6}$$

where, for convenience, s has the special value 5.

Any of the indices x, y may be zero.

Observe that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + 5 = p,$$
$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + 5 = q.$$

Obliterate the letters β, which do not occur in $\beta\alpha$-contacts, and the letters α, which do occur in $\beta\alpha$-contacts, there remains a succession

$$\alpha^{x_1}\beta\alpha^{x_2}\beta\alpha^{x_3}\beta\alpha^{x_4}\beta\alpha^{x_5}\beta\alpha^{x_6}$$

of p letters, β occurring s times.

Next obliterate in the original permutation the letters α, which do not occur in $\beta\alpha$-contacts, and the letters β, which do so occur; there remains a succession

$$\beta^{y_1}\alpha\beta^{y_2}\alpha\beta^{y_3}\alpha\beta^{y_4}\alpha\beta^{y_5}\alpha\beta^{y_6}$$

of q letters.

Take these two successions for the left and right portions of a new permutation, viz.:—

$$\alpha^{x_1}\beta\alpha^{x_2}\beta\alpha^{x_3}\beta\alpha^{x_4}\beta\alpha^{x_5}\beta\alpha^{x_6}\ .\ \beta^{y_1}\alpha\beta^{y_2}\alpha\beta^{y_3}\alpha\beta^{y_4}\alpha\beta^{y_5}\alpha\beta^{y_6}\alpha,$$

and we have made a perfectly definite transformation of a permutation involving exactly s $\beta\alpha$-contacts into another possessing the property that the letter β occurs s times in the first p places.

E.g., to transform

$$\beta^4\alpha^2\beta^3\alpha,$$

write it

$$\alpha^0\beta^3\quad \beta\alpha\quad \alpha\beta^2\quad \beta\alpha\,;$$

---

[*] This proof is of fundamental importance in the succeeding part of this investigation.

thence the successions

$$\alpha^0\beta\alpha\beta,$$

$$\beta^3\alpha\beta^2\alpha,$$

and the permutation

$$\beta\alpha\beta^4\alpha\beta^2\alpha,$$

in which β occurs twice in the first three places.

Now, it happens that these transformed permutations are very easily enumerated. The number of permutations of the letters in $\alpha^p\beta^q$, which possess the property that the letter β occurs exactly s times in the first p places is the coefficient of $\mu^s\alpha^p\beta^q$ in the development of

$$(\alpha + \mu\beta)^p (\alpha + \beta)^q.$$

This is evident from a consideration of the actual multiplication process.

Hence, in regard to the bipartite $\overline{pq}$,

$$(\alpha + \mu\beta)^p (\alpha + \beta)^q$$

is the generating function for the number of lines of route possessing a given number of essential nodes. The complete coefficient of $\alpha^p\beta^q$ is found to be

$$1 + \binom{p}{1}\binom{q}{1}\mu + \binom{p}{2}\binom{q}{2}\mu^2 + \ldots + \binom{p}{s}\binom{q}{s}\mu^s + \ldots + \binom{p}{q}\binom{q}{q}\mu^q,$$

generality not being lost by the supposition $p \nless q$. Hence the number of lines of route possessing s essential nodes is

$$\binom{p}{s}\binom{q}{s}.$$

Observe that the known formula

$$\binom{p}{0}\binom{q}{0} + \binom{p}{1}\binom{q}{1} + \ldots + \binom{p}{s}\binom{q}{s} + \ldots + \binom{p}{q}\binom{q}{q} = \binom{p+q}{p}$$

supplies a verification of the result.

It has also been proved that the number of compositions of $s + 1$ parts having all contacts positive-positive is $\binom{p}{s}\binom{q}{s}$.

25. On each of the $\binom{p}{s}\binom{q}{s}$ lines of route, which have s essential nodes, may be represented

$$2^{p+q-s-1}$$

compositions, because the $p+q-s-1$ non-essential nodes on each line of route may be selected as composition-nodes in this number of ways.

Hence, the total number of compositions is

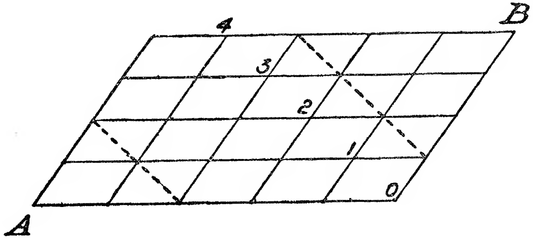
$$\mathrm{F}\,(pq) = \overset{s}{\Sigma} \binom{p}{s}\binom{q}{s} 2^{p+q-s-1}$$

(*cf.* the expressions for this number obtained in Arts. 12 and 14).

Analytically, this result is equivalent to the algebraic expansion

$$\frac{1}{1-2\,(x+y-xy)} = \frac{1}{(1-2x)\,(1-2y)} + \frac{2xy}{(1-2x)^2\,(1-2y)^2} + \frac{2^2x^2y^2}{(1-2x)^3\,(1-2y)^3} + \ldots$$

26. The important transformation of permutations established above is interesting when viewed graphically.



A line of route is traced by a certain succession of α and β segments. That portion of a line of route traced by the initial p segments terminates in one of the points 0, 1, 2, 3, 4 in the diagram. All these points lie on a straight line through the right-hand lower corner. If one of these points be marked s, s of the p segments will be β segments, and every line of route passing through the point s has the property that s symbols β occur in the first p places of the corresponding permutation of the symbols α and β. Hence, there is a one-to-one correspondence between the lines of route possessing 0, 1, 2, 3, 4, . . . s, . . . essential nodes and the lines of route passing through the points marked 0, 1, 2, 3, 4, . . . s, . . .

Observe that the number of lines of route in the graph, of which A and s are the initial and final points, is $\binom{p}{s}$, and in the graph of which s and B are the initial and final points, is $\binom{q}{s}$.

Hence of the graph AB the number of lines of route passing through the point s is

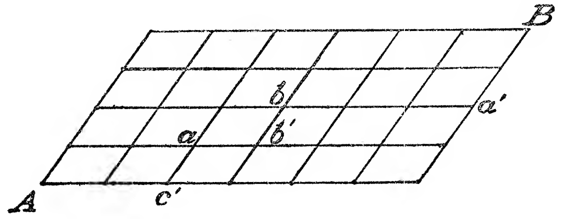
$$\binom{p}{s}\binom{q}{s},$$

and since every line of route must pass through one of the points 0, 1, 2, 3, 4, . . . s, . . . we have

$$\overset{s}{\Sigma}\binom{p}{s}\binom{q}{s} = \binom{p+q}{p}.^{*}$$

Inverse Bipartite Compositions.

27. A line of route being marked out on a reticulation from A to B, the inverse line of route is obtained by rotating the reticulation through two right angles and interchanging the letters A and B.



Consider a line of route from A to B having essential nodes a, b. On the inverse line of route a', b', c' are the essential nodes.

The principal compositions along these lines of route are—

$$(\overline{21}\ \overline{11}\ \overline{32}) \text{ from } A \text{ to } B.$$

$$(\overline{02}\ \overline{31}\ \overline{11}\ \overline{20}) \text{ from } B \text{ to } A.$$

---

[*] Consider also the points of the graph distant t segments from A. The number of such points is the coefficient of x^t in the product

$$(1 + x + x^2 + \ldots + x^p)\ (1 + x + x^2 + \ldots + x^q),$$

and if

$t < q + 1$ is equal to $t + 1$,
$> q < p + 1$,, $q + 1$,
$> p$,, $p + q - t + 1$.

These points lie on lines parallel to the line 01234 . . . s . . . and we obtain a graphical proof of the identities,

$$\binom{t}{0}\binom{p+q-t}{q} + \binom{t}{1}\binom{p+q-t}{q-1} + \ldots + \binom{t}{t}\binom{p+q-t}{q-t} = \binom{p+q}{p} \text{ for } t < q + 1;$$

$$\binom{t}{0}\binom{p+q-t}{q} + \binom{t}{1}\binom{p+q-t}{q-1} + \ldots + \binom{t}{q}\binom{p+q-t}{0} = \binom{p+q}{p} \text{ for } p + 1 > t > q;$$

$$\binom{t}{p}\binom{p+q-t}{0} + \binom{t}{p-1}\binom{p+q-t}{1} + \ldots + \binom{t}{t-q}\binom{p+q-t}{p+q-t} = \binom{p+q}{p} \text{ for } t > p$$

where the total number of terms in all these identities is

$$\sum_{0}^{q}(t+1) + (p-q)(q+1) + \sum_{t=p+1}^{t=p+q}(p+q-t+1) = (p+1)(q+1),$$

which is the total number of points in the graph (including A and B), and is therefore right.

All the contacts in these compositions are necessarily positive-positive; hence the leading and ending parts are the only ones that can involve a zero element; in the leading part the leading element may be zero, and in the ending part the ending element.

Every part, of a principal composition, which does not possess a zero element necessitates an essential node on the graph of the principal composition on the inverse line of route. Thus, if the one principal composition has s nodes, $s+1$ parts and t parts without a zero element, the other principal composition has t nodes, $t+1$ parts and s parts without a zero element.

These may be called "inverse principal compositions."

The number of pairs of inverse principal compositions is equal to half the number of distinct lines of route through the reticulation. The number of pairs is thus—

$$\tfrac{1}{2}\binom{p+q}{p}.$$

Otherwise, we may say that in regard to compositions, all of whose contacts are positive-positive, there is a one-to-one correspondence between those having $s+1$ parts and t parts without a zero element and those having $t+1$ parts and s parts without a zero element.

The number $s+1-t$ is either 0, 1, or 2.

E.g., for the bipartite $\overline{22}$ the correspondence is

$$\begin{array}{cc} s=0,\ t=1 & s=1,\ t=0 \\ (\overline{22}) & (\overline{02}\ \overline{20}) \\ & \\ s=1,\ t=1 & s=1,\ t=1 \\ (01\ \overline{21}) & (\overline{12}\ \overline{10}) \\ & \\ s=1,\ t=2 & s=2,\ t=1 \\ (\overline{11}\ \overline{11}) & (\overline{01}\ \overline{11}\ \overline{10}). \end{array}$$

28. This particular case of inversion leads easily to the general idea. Suppose a line route traced on a reticulation, and suppose marked the s essential nodes on the line drawn from A to B, and also the t essential nodes of the inverse line of route. These $s+t$ nodes are distinct. Along the line we may place an additional node at an unoccupied point and interpret the new compositions as read before and after rotation of the reticulation. The new compositions have each acquired an additional part; they each have the same number of parts without zero elements as before; the added node has either introduced a zero-positive contact into each or a positive-zero contact into each, according as the node is on an α or on a β line between adjacent nodes.

Altogether there are $p + q - 1 - s - t$ points at disposal, which may be selected as positions for nodes in $2^{p+q-1-s-t}$ ways. Suppose $j_1 + j_2$ additional nodes taken, such that j_1 zero-positive and j_2 positive-zero contacts are introduced into each composition; the number of parts in each composition will be increased by $j_1 + j_2$.

We have a one-to-one correspondence between the compositions, having

$s + 1 + j_1 + j_2$ parts,
t parts without a zero element,
j_1 zero-positive contacts,
j_2 positive-zero contacts;

and those having

$t + 1 + j_1 + j_2$ parts,
s parts without a zero element,
j_1 zero-positive contacts,
j_2 positive-zero contacts.

We have thus pairs of inverse compositions; the correspondence for the bipartite $\overline{22}$ is

$$\begin{array}{rcccc}
(s, t, j_1, j_2) = & (0, 1, 1, 0), & & (1, 0, 1, 0), & (1, 1, 0, 1), \\
& (\overline{10}\ \overline{12}) & \text{and} & (\overline{02}\ \overline{10}\ \overline{10}); & (\overline{11}\ \overline{01}\ \overline{10}); \\
& (0, 1, 0, 1), & & (1, 0, 0, 1), & (1, 1, 1, 0), \\
& (\overline{21}\ \overline{01}) & \text{and} & (\overline{01}\ \overline{01}\ \overline{20}); & (\overline{01}\ \overline{10}\ \overline{11}). \\
& (0, 1, 1, 1), & & (1, 0, 1, 1), & \\
& (\overline{10}\ \overline{11}\ \overline{01}) & \text{and} & (\overline{01}\ \overline{01}\ \overline{10}\ \overline{10}); &
\end{array}$$

Of the above, two compositions $(\overline{11}\ \overline{01}\ \overline{10})$, $(\overline{01}\ \overline{10}\ \overline{11})$ may be termed self-inverse.

29. There are twelve compositions, viz., those with zero-zero contacts, which do not appear in the theory.

To enumerate these compositions, consider the lines of route with s essential nodes or bends $\ulcorner$; every such line must have either $s - 1$, s, or $s + 1$ bends $\lrcorner$. If the allied $\alpha\beta$ permutation neither commences with α nor ends with β it is $s - 1$; if either α commences or β ends it is s; if both α commences and β ends it is $s + 1$.

The enumeration of the lines of route gives

$$\binom{p-1}{s-1}\binom{q-1}{s-1} \text{ for } s - 1 \text{ bends } \lrcorner\,;$$

$$\binom{p-1}{s}\binom{q-1}{s-1} + \binom{p-1}{s-1}\binom{q-1}{s} \text{ for } s \text{ bends } \lrcorner\,;$$

$$\binom{p-1}{s}\binom{q-1}{s} \text{ for } s + 1 \text{ bends } \lrcorner\,.$$

To obtain compositions along these lines of route which have not zero-zero contacts, we have at disposal

$$\begin{gathered} p+q-2s, \\ p+q-2s-1, \\ p+q-2s-2 \end{gathered}$$

points respectively at which non-essential nodes may be placed. Hence the number of compositions which have s positive-positive contacts (see Art. 19) and no zero-zero contacts is—

$$\begin{aligned} & 2^{p+q-2s}\binom{p-1}{s-1}\binom{q-1}{s-1} \\ + \; & 2^{p+q-2s-1}\left\{\binom{p-1}{s}\binom{q-1}{s-1}+\binom{p-1}{s-1}\binom{q-1}{s}\right\} \\ + \; & 2^{p+q-2s-2}\binom{p-1}{s}\binom{q-1}{s}, \end{aligned}$$

which is

$$\frac{(p+s)(q+s)}{pq}\,2^{p+q-2s-2}\binom{p}{s}\binom{q}{s}.$$

Hence the total number of compositions which have no zero-zero contacts is

$$2^{p+q-2}+\frac{(p+1)(q+1)}{pq}\,2^{p+q-4}\binom{p}{1}\binom{q}{1}+\frac{(p+2)(q+2)}{pq}\,2^{p+q-6}\binom{p}{2}\binom{q}{2}+\ldots,$$

and, since the total number of compositions which have s positive-positive contacts is

$$2^{p+q-1-s}\binom{p}{s}\binom{q}{s},$$

we find that the number of compositions having s positive-positive contacts and also zero-zero contacts is

$$2^{p+q-2s-2}\left\{2^{s+1}-\frac{(p+s)(q+s)}{pq}\right\}\binom{p}{s}\binom{q}{s},$$

and the total number of compositions having zero-zero contacts is

$$2^{p+q-2}+2^{p+q-4}\left(2^2-\frac{(p+1)(q+1)}{pq}\right)\binom{p}{1}\binom{q}{1}+2^{p+q-6}\left\{2^3-\frac{(p+2)(q+2)}{pq}\right\}\binom{p}{2}\binom{q}{2} \\ +\ldots$$

E.g., putting $\overline{pq}=\overline{22}$, we thus verify

$$4+(4-\tfrac{9}{4})\,4+\tfrac{1}{4}(8-4)=12$$

that the number is 12.

30. Self-inverse compositions can only occur upon self-inverse lines of route. For the existence of such a line, one at least of the elements of the bipartite number must be even. If both elements be even there is a central point in the reticulation, and the number of self-inverse lines of route for the bipartite $\overline{2p', 2q'}$ is equal to the number of lines of route of the bipartite $\overline{p'q'}$; that is to say, the number is

$$\binom{p'+q'}{p'}.$$

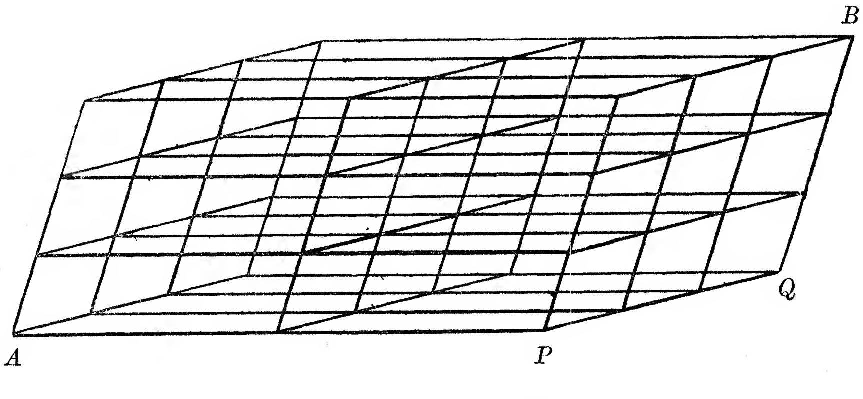
If the bipartite be $\overline{2p'+1, 2q'}$ it is easy to see that the number is also

$$\binom{p'+q'}{p'}.$$

The number of self-inverse compositions in both cases is evidently equal to the total number of compositions of the bipartite $\overline{p'q'}$.

§ 4. *The Graphical Representation of the Compositions of Tripartite and Multipartite Numbers.*

31. The graph of a tripartite number may be in either two or three dimensions. It may be derived from a bipartite graph in a manner similar to that in which the bipartite has been derived from the unipartite graph. In the tripartite number $\overline{pqr}$ we take $r+1$, exactly similar graphs of the bipartite $\overline{pq}$, and place them similarly with corresponding lines parallel, and like points lying on straight lines; when these straight lines are drawn the graph is complete. The $r+1$ bipartite graphs may be in the same plane or in parallel planes according as the tripartite graph is required to be in two or three dimensions.



The figure depicts the graph of the tripartite $\overline{233}$. The points of the reticulation are identical with the points of the $r+1$ reticulations of the bipartites $\overline{pq}$. Observe that in two dimensions there are intersections of lines which are not points of the reticulation. On the other hand in three dimensions all intersections are also points.

Other than the initial and final points A and B there are $(p+1)(q+1)(r+1)-2$ points.

The graph involves lines in three different directions; say an α, a β, and a γ direction. These are parallel to AP, PQ, and QB respectively.

Through each point of the graph pass lines in all three directions, and a segment joining two adjacent points is called an α segment when it is in the α direction.

A line of route proceeds from A to B, from point to point of the reticulation. The number of lines of route is the number of permutations of the symbols in $\alpha^p\beta^q\gamma^r$, and is, therefore,

$$\binom{p+q+r}{p,\, q,\, }.$$

A step along a line of route traverses in succession any number of α, β, and γ segments, and in any one step the segments must be taken in the order, α, β, γ. The number of segments traversed may be zero in one or two, but not in three of these directions.

A step is represented by a tripartite number $\overline{p_1q_1r_1}$ and a succession of steps, the first starting at A, and the last terminating at B, is represented by a succession of tripartite numbers constituting a composition of the tripartite $\overline{pqr}$.*

The graph of a composition is obtained by placing nodes at the points which terminate the first, second, &c., and penultimate steps.

Essential nodes occur upon lines of route whenever the direction at a point changes from β to α, from γ to α, or from γ to β. These will be alluded to briefly as $\beta\alpha$, $\gamma\alpha$, or $\gamma\beta$ essential nodes. To these essential nodes on the graph of a composition correspond respectively zero-positive, positive-positive, and positive-zero contacts in the composition itself. It is convenient to call these collectively essential contacts. The theory of conjugate composition exists as in the bipartite theory. On every line of route there are $p+q+r-1$ points, and if there be s essential nodes, $2^{p+q+r-1-s}$ distinct composition graphs can be delineated along the line of route. Each of these has s essential contacts, and, as in the bipartite case, we establish a one-to-one correspondence between the compositions having s essential contacts, and $s+t+1$ parts, and those having s essential contacts, and $p+q+r-t$ parts. The graph of a quadripartite number is derived from the tripartite graph, and generally the graphs of the multipartite numbers of order n from those of the multipartite numbers of order $n-1$, on the same principle as the tripartite from the bipartite; and, moreover, in two dimensions.

32. For the multipartite number

$$\overline{p_1p_2\ldots p_{n-1}p_n}$$

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* Observe that the thirteen compositions of the tripartite $\overline{111}$ are elegantly represented on the edges of a single cube, the six lines of route lying between opposite corners.

we take $p_n + 1$ exactly similar graphs of the multipartite number

$$\overline{p_1 p_2 \ldots p_{n-1}}$$

and place them similarly with corresponding lines parallel and like points lying on straight lines. When these straight lines are drawn the graph is complete. Other than the initial and final points A, B there are $(p_1 + 1)(p_2 + 1) \ldots (p_n + 1) - 2$ *points* in the graph.

There are lines in n different directions passing through or meeting at each point (including A and B). Call these directions the α_1 direction, the α_2, &c., the α_n direction. A segment joining two adjacent points is called an α segment when it is in the α direction. A step through the reticulation traverses in succession any number of $\alpha_1, \alpha_2, \ldots \alpha_n$ segments; but *in any one step* the segments must be taken in the order $\alpha_1, \alpha_2, \ldots \alpha_n$. In a step the number of steps traversed may be zero in any one, any two, &c., any $n - 1$ of the n different directions. If a step involve p'_1 segments in the α_1 direction, p'_2 segments in the α_2 direction and so on, it may be represented by the multipartite number

$$\overline{p'_1 p'_2 \ldots p'_n}.$$

A succession of steps, the first starting at A and the last terminating at B, is represented by a succession of multipartite numbers, constituting a composition of the multipartite

$$\overline{p_1 p_2 p_3 \ldots p_n}.$$

Any composition follows a certain line of route through the reticulation. The number of distinct lines of route is the number of permutations of the letters in

$$\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n},$$

and is therefore

$$\binom{p_1 + p_2 + \ldots + p_n}{p_1, p_2, \ldots p_{n-1}},$$

employing an obvious extension of notation.

The graph of a composition is obtained by placing nodes at the points which terminate the first, second, &c., and penultimate steps. Essential nodes occur upon a line of route whenever the direction at a point changes from α_u to α_t where $u > t$. At this point the contact between the adjacent parts of the composition is such that a part terminating with $n - u$ zero elements precedes a part commencing with $t - 1$ zero elements. We may speak of this as a contact of $n - u$ zeros with $t - 1$ zeros. The number of zeros in contact, being

$$n - 1 - (u - t),$$

may be any number from 0 to $n - 2$, according to the magnitude of $u - t$; and there are $j + 1$ different contacts for which the number of zeros in contact is j.

33. I now enquire, in respect of the tripartite reticulation, into the number of lines of route which possess exactly s_{21}, $\beta\alpha$ essential nodes, s_{32}, $\gamma\beta$ and s_{31}, $\gamma\alpha$ essential nodes. The second method of investigation adopted in the bipartite case will be employed.

Consider the permutations of the symbols in $\alpha^p\beta^q\gamma^r$ which possess s_{21}, $\beta\alpha$; s_{32}, $\gamma\beta$; and s_{31}, $\gamma\alpha$ contacts, and suppose

$$s_{21}+s_{32}+s_{31}=s.$$

It will be shown that the number of such permutations is equal to the number of permutations in which

β occurs s_{21} times in the first p places,
γ ,, s_{31} ,, ,, ,, ,,
γ ,, s_{32} ,, in the q places succeeding the first p.

A permutation of the former kind involves successions of letters

$$\beta\alpha,\ \gamma\beta,\ \gamma\alpha,\ \gamma\beta\alpha\,;$$

letters between consecutive successions being in alphabetical order.

As regards these successions, and attending to them alone, the permutation exhibits some permutation of the terms in

$$(\beta\alpha)^{s_{21}-\sigma}\ (\gamma\beta)^{s_{32}-\sigma}\ (\gamma\beta\alpha)^{\sigma}\ (\gamma\alpha)^{s_{31}}.$$

These permutations are

$$\frac{s_{21}+s_{32}+s_{31}-\sigma)\,!}{(s_{21}-\sigma)\,!\,(s_{32}-\sigma)\,!\,\sigma\,!\,s_{31}\,!}\ \text{in number.}$$

Selecting any one of these there are, with reference to the terms,

$$s_{21}+s_{32}+s_{31}-\sigma+1$$

different positions in which other letters may be placed. It is clear that α cannot be placed after a term $(\gamma\beta)$, or γ before a term $(\beta\alpha)$, for these placings would lead to additional $(\beta\alpha)$ and $(\gamma\beta)$ contacts. The letter α may be placed before any of the terms $(\beta\alpha)$, $(\gamma\beta)$, $(\gamma\beta\alpha)$, $(\gamma\alpha)$, and after any of the terms $(\beta\alpha)$, $(\gamma\beta\alpha)$, $(\gamma\alpha)$. Hence out of the $s_{21}+s_{32}+s_{31}-\sigma+1$ different positions (relative to the terms) $s_{21}+s_{31}+1$ positions may be occupied by a letter α. The letter β may be placed in all the $s_{21}+s_{32}+s_{31}-\sigma+1$ positions without the introduction of additional $\beta\alpha$, $\gamma\beta$, $\gamma\alpha$ terms. The letter γ must not be in a place preceding $(\beta\alpha)$, and hence it may occupy $s_{32}+s_{31}+1$ different positions.

Besides the letters occurring in the terms

$$(\beta\alpha)^{s_{21}-\sigma}\ (\gamma\beta)^{s_{32}-\sigma}\ (\gamma\beta\alpha)^{\sigma}\ (\gamma\alpha)^{s_{31}},$$

α occurs $p-s_{21}-s_{31}$ times,
β ,, $q-s_{21}-s_{32}+\sigma$ times,
γ ,, $r-s_{32}-s_{31}$ times.

The $p - s_{21} - s_{31}$ letters α may be distributed in $s_{21} + s_{31} + 1$ positions in

$$\binom{p}{s_{21} + s_{31}} \text{ ways.}$$

The $q - s_{21} - s_{32} + \sigma$ letters β may be distributed in $s_{21} + s_{32} + s_{31} - \sigma + 1$ positions in

$$\binom{q + s_{31}}{s_{21} + s_{32} + s_{31} - \sigma} \text{ ways.}$$

The $r - s_{32} - s_{31}$ letters γ may be distributed in $s_{32} + s_{31} + 1$ positions in

$$\binom{r}{s_{32} + s_{31}} \text{ ways.}$$

Hence, for any given permutation of the terms in

$$(\beta\alpha)^{s_{21}-\sigma} (\gamma\beta)^{s_{32}-\sigma} (\gamma\beta\alpha)^{\sigma} (\gamma\alpha)^{s_{31}}$$

there are

$$\binom{p}{s_{21} + s_{31}} \binom{q + s_{31}}{s_{21} + s_{32} + s_{31} - \sigma} \binom{r}{s_{32} + s_{31}}$$

arrangements of the remaining letters which do not introduce additional $\beta\alpha$, $\gamma\beta$ or $\gamma\alpha$ contacts.

Hence for a given value of σ there are

$$\binom{p}{s_{21} + s_{31}} \binom{q + s_{31}}{s_{21} + s_{32} + s_{31} - \sigma} \binom{r}{s_{32} + s_{31}} \frac{(s_{21} + s_{32} + s_{31} - \sigma)!}{(s_{21} - \sigma)!\,(s_{32} - \sigma)!\,\sigma!\,s_{31}!}$$

permutations.

To complete the enumeration we have to sum this expression in regard to σ.

We have

$$\begin{aligned}
&\Sigma \binom{q + s_{31}}{s_{21} + s_{32} + s_{31} - \sigma} \frac{(s_{21} + s_{32} + s_{31} - \sigma)!}{(s_{21} - \sigma)!\,(s_{32} - \sigma)!\,\sigma!\,s_{31}!} \\
&= \Sigma \frac{(q + s_{31})!}{(q - s_{21} - s_{32} + \sigma)!\,(s_{21} - \sigma)!\,(s_{32} - \sigma)!\,\sigma!\,s_{31}!} \\
&= \frac{(q + s_{31})!}{s_{31}!\,s_{32}!\,(q - s_{32})!} \Sigma \frac{s_{22}!}{\sigma!\,(s_{32} - \sigma)!} \cdot \frac{(q - s_{32})!}{(s_{21} - \sigma)!\,(q - s_{21} - s_{32} + \sigma)!} \\
&= \frac{(q + s_{31})!}{s_{31}!\,s_{32}!\,(q - s_{32})!} \Sigma \binom{s_{32}}{\sigma} \binom{q - s_{32}}{s_{21} - \sigma} \\
&= \frac{(q + s_{31})!}{s_{31}!\,s_{32}!\,(q - s_{32})!} \binom{q}{s_{21}} \\
&= \binom{s_{21} + s_{31}}{s_{21}} \binom{q}{s_{32}} \binom{q + s_{31}}{s_{21} + s_{31}}.
\end{aligned}$$

Hence the number of permutations sought is

$$\binom{s_{21}+s_{31}}{s_{21}}\binom{p}{s_{21}+s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{21}+s_{31}}\binom{r}{s_{32}+s_{31}}.$$

34. It can be shown by direct expansion that this number is the coefficient of $\lambda_{21}{}^{s_{21}}\lambda_{31}{}^{s_{31}}\lambda_{32}{}^{s_{32}}\alpha^p\beta^q\gamma^r$ in the development of

$$(\alpha+\lambda_{21}\beta+\lambda_{31}\gamma)^p\,(\alpha+\beta+\lambda_{32}\gamma)^q\,(\alpha+\beta+\gamma)^r,$$

which expresses the number of permutations in which

β occurs s_{21} times in the first p places.
γ ,, s_{31} ,, ,,
γ ,, s_{32} times in the q places succeeding the first p.

This remarkable identity of numbers suggests a one-to-one correspondence between the permutations of the two kinds, but I have not as yet been able to determine the law of the transformation. It would appear to be a very difficult problem. The number of lines of route in the tripartite reticulation which possess

s_{21}, $\beta\alpha$ essential nodes,
s_{32}, $\gamma\beta$,, ,,
s_{31}, $\gamma\alpha$,, ,,

is thus

$$\binom{s_{21}+s_{31}}{s_{21}}\binom{p}{s_{21}+s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{21}+31}\binom{r}{s_{32}+s_{31}}.$$

35. Hence the identity

$$\frac{p+q+r)!}{p!\,q!\,r!}=\Sigma\binom{s_{21}+s_{31}}{s_{21}}\binom{p}{s_{21}+s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{21}+s_{31}}\binom{r}{s_{32}+s_{31}},$$

the summation being for all positive integral, including zero, values of s_{21}, s_{31}, s_{32}, which yield positive terms.

36. The generating function for the number of lines of route having s essential nodes; that is $s_{21}+s_{31}+s_{32}=s$; is

$$(\alpha+\lambda\beta+\lambda\gamma)^p\,(\alpha+\beta+\lambda\gamma)^q\,(\alpha+\beta+\gamma)^r,$$

the number in question being the coefficient of $\lambda^s\alpha^p\beta^q\gamma^r$.

37. The whole coefficient of $\alpha^p\beta^q\gamma^r$ being

$$C_0 + C_1\lambda + C_2\lambda^2 + \ldots + C_s\lambda^s + \ldots$$

$$C_s = \Sigma \binom{s_{21} + s_{31}}{s_{21}} \binom{p}{s_{21} + s_{31}} \binom{q}{s_{32}} \binom{q + s_{31}}{s_{21} + s_{31}} \binom{r}{s_{32} + s_{31}}$$

the summation being subject to the condition

$$s_{21} + s_{31} + s_{32} = s.$$

Moreover, denoting the whole number of compositions of the tripartite $\overline{pqr}$ by F (pqr), we have

$$\begin{aligned} F(pqr) &= \overset{s}{\Sigma}\, C_s\, 2^{p+q+r-s-1} \\ &= 2^{p+q+r-1} \overset{s}{\Sigma}\, C_s \left(\tfrac{1}{2}\right)^s. \end{aligned}$$

38. Hence F (pqr) is the coefficient of $\alpha^p\beta^q\gamma^r$ in the development of the product

$$2^{p+q+r-1} (\alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma)^p (\alpha + \beta + \tfrac{1}{2}\gamma)^q (\alpha + \beta + \gamma)^r,$$

which is more conveniently written

$$\tfrac{1}{2} (2\alpha + \beta + \gamma)^p (2\alpha + 2\beta + \gamma)^q (2\alpha + 2\beta + 2\gamma)^r.$$

39. This product is a generating function which enumerates the compositions of the single multipartite number $\overline{pqr}$, but the generating function of all tripartite numbers can be at once derived from it.

It is

$$\frac{1}{2} \frac{1}{\{1 - s(2\alpha + \beta + \gamma)\}\{1 - t(2\alpha + 2\beta + \gamma)\}\{1 - u(2\alpha + 2\beta + 2\gamma)\}},$$

in which, when expanded, the coefficient of $(s\alpha)^p(t\beta)^q(u\gamma)^r$ is the number of compositions of the tripartite number $\overline{pqr}$.

The generating function previously obtained for tripartite numbers from the analytical theory was

$$\frac{\alpha + \beta + \gamma - \beta\gamma - \gamma\alpha - \alpha\beta + \alpha\beta\gamma}{1 - 2(\alpha + \beta + \gamma - \beta\gamma - \gamma\alpha - \alpha\beta + \alpha\beta\gamma)},$$

the number of compositions being given by the coefficient of $\alpha^p\beta^q\gamma^r$. The addition of $\frac{1}{2}$ to this fraction brings it to the better form

$$\frac{1}{2} \frac{1}{1 - 2(\alpha + \beta + \gamma - \beta\gamma - \gamma\alpha - \alpha\beta + \alpha\beta\gamma)},$$

which is consistent also with the circumstance that it was found convenient analytically to regard the number of compositions of multipartite zero as being the fraction $\frac{1}{2}$.

40. The number may be stated as the coefficient of $(s\alpha)^p(t\beta)^q(u\gamma)^r$ in the expansion of

$$\frac{1}{2}\frac{1}{1-2(s\alpha+t\beta+u\gamma-tu\beta\gamma-us\gamma\alpha-st\alpha\beta+stu\alpha\beta\gamma)},$$

and we have shown that this fraction is equivalent to that portion of the expansion of the fraction

$$\frac{1}{2}\frac{1}{\{1-s(2\alpha+\beta+\gamma)\}\{1-t(2\alpha+2\beta+\gamma)\}\{1-u(2\alpha+2\beta+2\gamma)\}}$$

which is a function only of $s\alpha$, $t\beta$, and $u\gamma$.

41. It will now be shown that the fraction

$$\frac{1}{2}\frac{1}{\{1-s_1(2\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(2\alpha_1+2\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)\}}$$

is, in fact, a generating function which enumerates the compositions of multipartite numbers of order n.

This important theorem will be demonstrated by showing that the aggregate of terms in the expansion of the last written fraction, which is composed entirely of powers of $s_1\alpha_1$, $s_2\alpha_2$, . . . $s_n\alpha_n$, is correctly represented by the fraction

$$\frac{1}{2}\frac{1}{1-2(\Sigma s_1\alpha_1-\Sigma s_1s_2\alpha_1\alpha_2+\ldots(-)^{n+1}s_1s_2\ldots s_n\alpha_1\alpha_2\ldots\alpha_n)},$$

which has been already shown to be a true generating function.

42. For brevity put

$$S_\kappa=s_\kappa(2\alpha_1+2\alpha_2+\ldots+2\alpha_\kappa+\alpha_{\kappa+1}+\ldots+\alpha_n)=s_\kappa(A_\kappa+2\alpha_\kappa)$$
$$b_1=\Sigma s_1\alpha_1,\ b_2=\Sigma s_1s_2\alpha_1\alpha_2,\ \&c.\ldots$$
$$M=(1-2s_1\alpha_1)(1-2s_2\alpha_2)\ldots(1-2s_n\alpha_n).$$

The two fractions under comparison are

$$\frac{1}{2}\frac{1}{-1+2(1-s_1\alpha_1)(1-s_2\alpha_2)\ldots(1-s_n\alpha_n)}=\frac{1}{2N},$$

and

$$\frac{1}{2}\frac{1}{(1-S_1)(1-S_2)\ldots(1-S_n)}=\frac{1}{2D}.$$

Since $\mathrm{M} - \mathrm{N} = (2^2 - 2)\,b_2 - (2^3 - 2)\,b_3 + \ldots + (-)^n\,(2^n - 2)b_n$,

$$\frac{\mathrm{N}}{\mathrm{D}} = \frac{\mathrm{M}}{\mathrm{D}} - \frac{\mathrm{M}-\mathrm{N}}{\mathrm{D}} = \prod_1^n \frac{1 - 2s_\kappa\alpha_\kappa}{1 - \mathrm{S}_\kappa} - \frac{(2^2-2)\,b_2 - (2^3-2)\,b_3 + \ldots + (-)^n\,(2^n-2)\,b_n}{(1-\mathrm{S}_1)\,(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)}$$

$$= \prod_1^n \left(1 + \frac{s_\kappa \mathrm{A}_\kappa}{1 - \mathrm{S}_\kappa}\right) - \frac{(2^2-2)\,b_2 - \ldots + (-)^n\,(2^n-2)\,b_n}{(1-\mathrm{S}_1)\,(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)}\,;$$

the second member of the right-hand side of this identity has now to be transformed into a series of partial fractions of the same form as those which arise from the product

$$\prod_1^n \left(1 + \frac{s_\kappa \mathrm{A}_\kappa}{1 - \mathrm{S}_\kappa}\right).$$

43. Let $\kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_t$ be any t different numbers selected from the first n integers $1, 2, \ldots n$, arranged in ascending order of magnitude.

Let

$$\mathrm{B}^{(1)}_{\kappa_1\kappa_2} = 2s_{\kappa_1}\alpha_{\kappa_1}s_{\kappa_2}\alpha_{\kappa_2}$$

$$\mathrm{B}^{(1)}_{\kappa_1\kappa_2\kappa_3} = \mathrm{B}^{(1)}_{\kappa_2\kappa_3}\mathrm{S}_{\kappa_1} + \mathrm{B}^{(1)}_{\kappa_3\kappa_1}\mathrm{S}_{\kappa_2} + \mathrm{B}^{(1)}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} = \Sigma\,\mathrm{B}^{(1)}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\ldots\mathrm{S}_{\kappa_t},$$

the summation being for the $\binom{t}{2}$ terms obtained from the $\binom{t}{2}$ expressions $\mathrm{B}_{\kappa_1\kappa_2}{}^{(1)}$ that it is possible to construct from the t integers $\kappa_1, \kappa_2, \ldots \kappa_t$.

Further, let

$$\mathrm{B}^{(j)}_{\kappa_1\kappa_2\ldots\kappa_{j+1}} = (2^{j+1} - 2)\,s_{\kappa_1}\alpha_{\kappa_1}s_{\kappa_2}\alpha_{\kappa_2}\ldots s_{\kappa_{j+1}}\,\alpha_{\kappa_{j+1}}$$

$$\mathrm{B}^{(j)}_{\kappa_1\kappa_2\ldots\kappa_t} = \Sigma\,\mathrm{B}^{(j)}_{\kappa_1\kappa_2\ldots\kappa_{j+1}}\mathrm{S}_{j+2}\ldots\mathrm{S}_t$$

the summation being for the $\binom{t}{j+1}$ terms obtained from the $\binom{t}{j+1}$ expressions $\mathrm{B}_{\kappa_1\kappa_2\ldots\kappa_{j+1}}{}^{(j)}$ that it is possible to construct from the t integers $\kappa_1, \kappa_2, \ldots \kappa_t$.

44. The following lemma is required :—

"*Lemma.*

$$\Sigma\,\mathrm{B}^{(j)}_{\kappa_1\kappa_2\ldots\kappa_{t-1}}\mathrm{S}_{\kappa_t} = \binom{t-j-1}{1}\mathrm{B}^{(j)}_{\kappa_1\kappa_2\ldots\kappa_t},$$

$$\Sigma\,\mathrm{B}^{(j)}_{\kappa_1\kappa_2\ldots\kappa_{t-2}}\mathrm{S}_{\kappa_{t-1}}\mathrm{S}_{\kappa_t} = \binom{t-j-1}{2}\mathrm{B}^{(j)}_{\kappa_1\kappa_2\ldots\kappa_t},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\Sigma\,\mathrm{B}^{(j)}_{\kappa_1\kappa_2\ldots\kappa_{t-s}}\mathrm{S}_{\kappa_{t-s+1}}\ldots\mathrm{S}_{\kappa_t} = \binom{t-j-1}{s}\mathrm{B}^{(j)}_{\kappa_1\kappa_2\ldots\kappa_t}$$

the summations being subject to the same conditions as before. The truth of these relations is obvious from the elementary theory of permutations."

45. Consider now the series of functions—

$$\begin{aligned}
&\mathrm{P}_{\kappa_1\kappa_2} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2},\\
&\mathrm{P}_{\kappa_1\kappa_2\kappa_3} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2\kappa_3} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\kappa_3},\\
&\mathrm{P}_{\kappa_1\kappa_2\kappa_3\kappa_4} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2\kappa_3\kappa_4} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\kappa_3\kappa_4} + \mathrm{B}^{(3)}_{\kappa_1\kappa_2\kappa_3\kappa_4},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t} + \ldots + (-)^t\,\mathrm{B}^{(t-1)}_{\kappa_1\kappa_2\ldots\kappa_t}
\end{aligned}$$

From the foregoing lemma are derived the relations—

$$\begin{aligned}
&\Sigma\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\ldots\mathrm{S}_{\kappa_t} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t},\\
&\Sigma\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3}\mathrm{S}_{\kappa_4}\ldots\mathrm{S}_{\kappa_t} = \binom{t-2}{1}\mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t},\\
&\Sigma\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3\kappa_4}\mathrm{S}_{\kappa_5}\ldots\mathrm{S}_{\kappa_t} = \binom{t-2}{2}\mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \binom{t-3}{1}\mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t} + \mathrm{B}^{(3)}_{\kappa_1\kappa_2\ldots\kappa_t},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\Sigma\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_s}\mathrm{S}_{\kappa_{s+1}}\ldots\mathrm{S}_{\kappa_t} = \binom{t-2}{s-2}\mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \binom{t-3}{s-3}\mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t} + \ldots + (-)^s\,\mathrm{B}^{(s-1)}_{\kappa_1\kappa_2\ldots\kappa_t}.\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Hence, by addition with alternate signs,

$$\begin{aligned}
(-)^t\,\mathrm{B}_{\kappa_1\kappa_2\ldots\kappa_t}{}^{(t-1)} = \mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t} - \Sigma\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{t-1}}\mathrm{S}_{\kappa_t} + \Sigma\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{t-2}}\mathrm{S}_{\kappa_{t-1}}\mathrm{S}_{\kappa_t} - \ldots\\
+ (-)^t\,\Sigma\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_4}\ldots\mathrm{S}_{\kappa_t}.
\end{aligned}$$

Here the summations are in respect of the t numbers $\kappa_1, \kappa_2, \ldots \kappa_t$.

46. A similar identity may be established for every selection of t integers out of the numbers 1, 2, 3, . . . n.

Summing all these identities, and remembering that

$$\overset{n}{\Sigma}\,\mathrm{B}_{\kappa_1\kappa_2\ldots\kappa_t}{}^{(t-1)} = (2^t - 2)\,b_t,$$

we obtain

$$(-)^t(2^t - 2)\,b_t = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t} - \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{t-1}}\mathrm{S}_{\kappa_t} + \ldots + (-)^t\,\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\ldots\mathrm{S}_{\kappa_t},$$

or, attributing to t all integer values from 2 to n inclusive, we have the series of relations :—

$$(2^2 - 2)\,b_2 = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}$$

$$-\,(2^3 - 2)\,b_3 = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3} - \Sigma\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}$$

$$+\,(2^4 - 2)\,b_4 = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3\kappa_4} - \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3}\mathrm{S}_{\kappa_4} + \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_4}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$(-)^n\,(2^n - 2)\,b_n = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_n} - \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{n-1}}\mathrm{S}_{\kappa_n} + \ldots + (-)^n\,\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\ldots\mathrm{S}_{\kappa_n}.$$

Now

$$\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3} = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}(\mathrm{S}_{\kappa_3} + \mathrm{S}_{\kappa_4} + \ldots + \mathrm{S}_{\kappa_n})$$

$$\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_4} = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}(\mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_4} + \mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_5} + \ldots + \mathrm{S}_{\kappa_{n-1}}\mathrm{S}_{\kappa_n})$$

with similar relations in the case of the other symmetric functions.

47. Hence, by addition,

$$\begin{aligned}
&(2^2 - 2)\,b_2 - (2^3 - 2)\,b_3 + \ldots + (-)^n\,(2^n - 2)\,b_n \\
&\quad = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\,(1 - \mathrm{S}_{\kappa_3})\,(1 - \mathrm{S}_{\kappa_4})\ldots(1 - \mathrm{S}_{\kappa_n}) \\
&\qquad + \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3}\,(1 - \mathrm{S}_{\kappa_4})\,(1 - \mathrm{S}_{\kappa_5})\ldots(1 - \mathrm{S}_{\kappa_n}) \\
&\qquad + \ldots \\
&\qquad + \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_1\ldots\kappa_{n-1}}(1 - \mathrm{S}_{\kappa_n}) \\
&\qquad + \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_n},
\end{aligned}$$

and thence,

$$\begin{aligned}
&\frac{(2^2 - 2)\,b_2 - (2^3 - 2)\,b_3 + \ldots + (-)^n\,(2^n - 2)\,b_n}{(1 - \mathrm{S}_1)\,(1 - \mathrm{S}_2)\ldots(1 - \mathrm{S}_n)} \\
&\quad = \overset{n}{\Sigma}\,\frac{\mathrm{P}_{\kappa_1\kappa_2}}{(1 - \mathrm{S}_{\kappa_1})\,(1 - \mathrm{S}_{\kappa_2})} \\
&\qquad + \overset{n}{\Sigma}\,\frac{\mathrm{P}_{\kappa_1\kappa_2\kappa_3}}{(1 - \mathrm{S}_{\kappa_1})\,(1 - \mathrm{S}_{\kappa_2})\,(1 - \mathrm{S}_{\kappa_3})} \\
&\qquad + \ldots \\
&\qquad + \overset{n}{\Sigma}\,\frac{\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t}}{(1 - \mathrm{S}_{\kappa_1})\,(1 - \mathrm{S}_{\kappa_2})\ldots(1 - \mathrm{S}_{\kappa_t})} \\
&\qquad + \ldots \\
&\qquad + \frac{\mathrm{P}_{123\ldots n}}{(1 - \mathrm{S}_1)\,(1 - \mathrm{S}_2)\ldots(1 - \mathrm{S}_n)}.
\end{aligned}$$

The relation

$$\frac{\mathrm{N}}{\mathrm{D}} = \prod_{1}^{n}\left(1 + \frac{s_\kappa \mathrm{A}_\kappa}{1 - \mathrm{S}_\kappa}\right) - \frac{(2^2-2)\,b_2 - (2^3-2)\,b_3 + \ldots + (-)^n (2^n - 2)\,b_n}{(1-\mathrm{S}_1)(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)}$$

now becomes

$$\frac{\mathrm{N}}{\mathrm{D}} = 1 + \sum^{n} \frac{s_{\kappa_1}\mathrm{A}_{\kappa_1}}{1-\mathrm{S}_{\kappa_1}} + \sum^{n} \frac{s_{\kappa_1}s_{\kappa_2}\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2} - \mathrm{P}_{\kappa_1\kappa_2}}{(1-\mathrm{S}_{\kappa_1})(1-\mathrm{S}_{\kappa_2})} + \ldots + \sum^{n} \frac{s_{\kappa_1}\ldots s_{\kappa_{n-1}}\mathrm{A}_{\kappa_1}\ldots\mathrm{A}_{\kappa_{n-1}} - \mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{n-1}}}{(1-\mathrm{S}_{\kappa_1})(1-\mathrm{S}_{\kappa_1})\ldots(1-\mathrm{S}_{\kappa_{n-1}})},$$

the final fraction

$$\frac{s_{\kappa_1}s_{\kappa_2}\ldots s_{\kappa_n}\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2}\ldots\mathrm{A}_{\kappa_{n-1}} - \mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_n}}{(1-\mathrm{S}_1)(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)}$$

vanishing as will be seen presently.

48. This form of the fraction $\frac{\mathrm{N}}{\mathrm{D}}$ will be employed in order to show that the expansions of $\frac{1}{\mathrm{D}}$ and $\frac{1}{\mathrm{N}}$ are effectively identical. This arises from the circumstance that the right-hand side of the above identity when expanded contains no single term expressible as a function of $s_1\alpha_1, s_2\alpha_2, s_3\alpha_3, \ldots s_n\alpha_n$.

49. Of the fractions $\Sigma \frac{s_{\kappa_1}\mathrm{A}_{\kappa_1}}{1-\mathrm{S}_{\kappa_1}}$ consider the typical fraction

$$\frac{s_{\kappa_1}\mathrm{A}_{\kappa_1}}{1-\mathrm{S}_{\kappa_1}} = \frac{s_{\kappa_1}(2\alpha_1 + \ldots + 2\alpha_{\kappa_1-1} + \alpha_{\kappa_1+1} + \ldots + \alpha_n)}{1 - s_{\kappa_1}(2\alpha_1 + \ldots + 2\alpha_{\kappa_1} + \alpha_{\kappa_1+1} + \ldots + \alpha_n)};$$

since the numerator contains a factor s_{κ_1} and no quantity α_{κ_1} the fraction when expanded can contain no term which is a function of $s_1\alpha_1, s_2\alpha_2, \ldots s_n\alpha_n$ alone.

Consider next the typical fraction

$$\frac{s_{\kappa_1}s_{\kappa_2}\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2} - \mathrm{P}_{\kappa_1\kappa_2}}{(1-\mathrm{S}_{\kappa_1})(1-\mathrm{S}_{\kappa_2})};$$

the numerator is

$$s_{\kappa_1}s_{\kappa_2}(\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2} - 2\alpha_{\kappa_1}\alpha_{\kappa_2})$$

and

$$\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2} = (2\alpha_1 + \ldots + 2\alpha_{\kappa_1-1} + \ldots + \alpha_{\kappa_2} + \ldots + \alpha_n)(2\alpha_1 + \ldots + 2\alpha_{\kappa_1} + \ldots + 2\alpha_{\kappa_2-1} + \ldots + \alpha_n),$$

and, therefore, contains a term

$$2\alpha_{\kappa_1}\alpha_{\kappa_2};$$

hence the numerator is free from such a term and every term in it contains one of the quantities

$$\alpha_{\kappa_3}, \alpha_{\kappa_4}, \ldots \alpha_{\kappa_n};$$

but the whole fraction contains no quantities

$$s_{\kappa_3}, \alpha_{\kappa_4}, \ldots s_{\kappa_n}$$

and thus it is manifest that the fraction can contribute no term which is a function of

$$s_1\alpha_1, \ldots s_n\alpha_n.$$

50. To simplify the discussion of the remaining fractions it is necessary to consider some particular properties of the typical numerator.

The function

$$\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t}$$

may be expressed as

$$\mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t} + \ldots + (-)^t\,\mathrm{B}^{(t-1)}_{\kappa_1\kappa_2\ldots\kappa_t},$$

or as

$$\begin{aligned}\{&\overset{t}{\Sigma}(2^2-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}(\mathrm{A}_{\kappa_3}+2\alpha_{\kappa_3})\ldots(\mathrm{A}_{\kappa_t}+2\alpha_{\kappa_t})\\ &-\overset{t}{\Sigma}(2^3-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\alpha_{\kappa_3}(\mathrm{A}_{\kappa_4}+2\alpha_{\kappa_4})\ldots(\mathrm{A}_{\kappa_t}+2\alpha_{\kappa_t})\\ &+\ldots\\ &+(-)^t(2^t-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\ldots\alpha_{\kappa_t}\}s_{\kappa_1}s_{\kappa_2}\ldots s_{\kappa_t}\end{aligned}$$

the summations being in respect of the t numbers

$$\kappa_1, \kappa_2, \ldots \kappa_t.$$

Writing

$$\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t} = \mathrm{P}'_{\kappa_1\kappa_2\ldots\kappa}\; s_{\kappa_1}s_{\kappa_2}\ldots s_{\kappa_t},$$

we have now the identity

$$\begin{aligned}\mathrm{P}'_{\kappa_1\kappa_2\ldots\kappa_t} = \overset{t}{\Sigma}(2^2-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\mathrm{A}_{\kappa_3}\mathrm{A}_{\kappa_4}\ldots\mathrm{A}_{\kappa_t} + \Sigma(2^3-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\alpha_{\kappa_3}\mathrm{A}_{\kappa_4}\ldots\mathrm{A}_{\kappa_t}\\ +\ldots+(2^t-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\ldots\alpha_{\kappa}.\end{aligned}$$

51. To establish this it is sufficient to observe that the coefficient of

$$\alpha_{\kappa_1}\alpha_{\kappa_2}\ldots\alpha_{\kappa_s}\mathrm{A}_{\kappa_{s+1}}\ldots\mathrm{A}_{\kappa_t}$$

in the former series is

$$\binom{s}{2} 2^{s-2} (2^2 - 2) - \binom{s}{3} 2^{s-3} (2^3 - 2) + \ldots + (-)^{s+1} \binom{s}{s} (2^s - 2),$$

which has the value $2^s - 2$.

52. Hence the numerator of the typical fraction may, to a factor $s_{\kappa_1} s_{\kappa_2} \ldots s_{\kappa_t}$ *près* be thrown into the form :—

$$2 (A_{\kappa_1} + \alpha_{\kappa_1}) (A_{\kappa_2} + \alpha_{\kappa_2}) \ldots (A_{\kappa} + \alpha_{\kappa_t}) - (A_{\kappa_1} + 2\alpha_{\kappa_1}) (A_{\kappa_2} + 2\alpha_{\kappa_2}) \ldots (A_{\kappa_t} + 2\alpha_{\kappa_t}),$$

which is of great service.

Of the series of quantities

$$s_1, s_2, \ldots s_n,$$

the typical fraction contains only the t quantities

$$s_{\kappa_1}, s_{\kappa_2}, \ldots s_{\kappa_t}.$$

If it can be shown that the numerator consists of terms each of which contains quantities of the series

$$\alpha_1, \alpha_2, \ldots \alpha_n,$$

which are not included in the set

$$\alpha_{\kappa_1}, \alpha_{\kappa_2}, \ldots \alpha_{\kappa_t},$$

it will follow at once that the fraction can, when expanded, contain no term which is a function of

$$s_1\alpha_1, s_2\alpha_2, \ldots s_n\alpha_n.$$

It suffices to show that the numerator vanishes when all the quantities of the set

$$\alpha_1, \alpha_2, \ldots \alpha_n,$$

not included in the set

$$\alpha_{\kappa_1}, \alpha_{\kappa_2}, \ldots \alpha_{\kappa_t},$$

are put equal to zero.

Under these circumstances, remembering that $\kappa_1\ \kappa_2 \ldots \kappa_n$ are in ascending order of magnitude, we have

$$\begin{aligned}
A_{\kappa_1} &= \alpha_{\kappa_2} + \alpha_{\kappa_3} + \ldots + \alpha_{\kappa_t} \\
A_{\kappa_2} &= 2\alpha_{\kappa_1} + \alpha_{\kappa_3} + \ldots + \alpha_{\kappa_t} \\
A_{\kappa_3} &= 2\alpha_{\kappa_1} + 2\alpha_{\kappa_2} + \alpha_{\kappa_4} + \ldots + \alpha_{\kappa_t}. \\
&\ldots\ldots\ldots\ldots\ldots \\
A_{\kappa_{t-1}} &= 2 (\alpha_{\kappa_1} + \alpha_{\kappa_2} + \ldots + \alpha_{\kappa_{t-2}}) + \alpha_{\kappa_t} \\
A_{\kappa_t} &= 2 (\alpha_{\kappa_1} + \alpha_{\kappa_2} + \ldots + \alpha_{\kappa_{t-1}})
\end{aligned}$$

and the relations

$$\begin{aligned} A_{\kappa_1} + 2\alpha_{\kappa_1} &= A_{\kappa_2} + \alpha_{\kappa_2} \\ A_{\kappa_2} + 2\alpha_{\kappa_2} &= A_{\kappa_3} + \alpha_{\kappa_3} \\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ A_{\kappa_{t-1}} + 2\alpha_{\kappa_{t-1}} &= A_{\kappa_t} + \alpha_{\kappa_t} \\ A_{\kappa_t} + 2\alpha_{\kappa_t} &= 2\,(A_{\kappa_1} + \alpha_{\kappa_1}) \end{aligned}$$

hence the numerator factor becomes

$$2\,(A_{\kappa_1} + \alpha_{\kappa_1})\,(A_{\kappa_2} + \alpha_{\kappa_2}) \ldots (A_{\kappa_t} + \alpha_{\kappa_t}) - (A_{\kappa_2} + \alpha_{\kappa_2}) \ldots (A_{\kappa_t} + \alpha_{\kappa_t})\, 2\,(A_{\kappa_1} + \alpha_{\kappa_1}) = 0.$$

53. This result proves incidentally, as stated above, that the final fraction

$$\frac{s_{\kappa_1}\, s_{\kappa_2} \ldots s_{\kappa_n}\, A_{\kappa_1}\, A_{\kappa_2} \ldots A_{\kappa_n} - P_{\kappa_1\,\kappa_2 \ldots \kappa_n}}{(1 - S_1)\,(1 - S_2) \ldots (1 - S_n)}$$

vanishes identically.

It also terminates the proof which it has been the object of this portion of the investigation to set forth.

54. The analytical result may be stated as follows :—

The fraction

$$\frac{1}{2}\,\frac{1}{\{1 - s_1\,(2\alpha_1 + \alpha_2 + \ldots + \alpha_n)\}\,\{1 - s_2\,(2\alpha_1 + 2\alpha_2 + \ldots + \alpha_n)\} \ldots \{1 - s_n\,(2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_\kappa)\}}$$

is equal to the product of the fraction

$$\frac{1}{2}\,\frac{1}{1 - 2\,(\Sigma\, s_1\alpha_1 - \Sigma\, s_1 s_2 \alpha_1 \alpha_2 + \ldots + (-)^{n+1}\, s_1 s_2 \ldots s_n\, \alpha_1 \alpha_2 \ldots \alpha_n)}$$

and the series

$$1 + \Sigma\, \frac{2\,(A_{\kappa_1} + \alpha_{\kappa_1})\,(A_{\kappa_2} + \alpha_{\kappa_2}) \ldots (A_{\kappa_t} + \alpha_{\kappa_t}) - (A_{\kappa_1} + 2\alpha_{\kappa_1})\,(A_{\kappa_2} + 2\alpha_{\kappa_2}) \ldots (A_{\kappa_t} + 2\alpha_{\kappa_t})}{(1 - S_{\kappa_1})\,(1 - S_{\kappa_2}) \ldots (1 - S_{\kappa_t})}$$

where

$$S_\kappa = s_\kappa\,(2\alpha_1 + \ldots + 2\alpha_\kappa + \alpha_{\kappa+1} + \ldots + \alpha_n) = s_\kappa\,(A_\kappa + 2\alpha_\kappa)$$

and the summation is in regard to every selection of t integers from the series 1, 2, 3, ... n, and t takes all values from 1 to $n - 1$.

55. It may be interesting to give the simplest cases at length.

Order 2.

$$\frac{1}{2}\,\frac{1}{\{1 - s_1\,(2\alpha_1 + \alpha_2)\}\,\{1 - s_2\,(2\alpha_1 + 2\alpha_2)\}}$$

$$= \frac{1}{2}\,\frac{1}{1 - 2\,(s_1\alpha_1 + s_2\alpha_2 - s_1 s_2 \alpha_1 \alpha_2)} \times \left[1 + \frac{s_1\alpha_2}{1 - s_1\,(2\alpha_1 + \alpha_2)} + \frac{2\, s_2\alpha_1}{1 - s_2\,(2\alpha_1 + 2\alpha_2)}\right].$$

56. *Order* 3.

$$\tfrac{1}{2}\frac{1}{\{1-s_1(2\alpha_1+\alpha_2+\alpha_3)\}\{1-s_2(2\alpha_1+2\alpha_2+\alpha_3)\}\{1-s_3(2\alpha_1+2\alpha_2+2\alpha_3)\}}$$

$$=\tfrac{1}{2}\frac{1}{1-2(s_1\alpha_1+s_2\alpha_2+s_3\alpha_3-s_2s_3\alpha_2\alpha_3-s_3s_1\alpha_3\alpha_1-s_1s_2\alpha_1\alpha_2+s_1s_2s_3\alpha_1\alpha_2\alpha_3)}$$

$$\times\left[1+\frac{s_1(\alpha_2+\alpha_3)}{1-s_1(2\alpha_1+\alpha_2+\alpha_3)}+\frac{s_2(2\alpha_1+\alpha_3)}{1-s_2(2\alpha_1+2\alpha_2+\alpha_3)}+\frac{s_3(2\alpha_1+2\alpha_2)}{1-s_3(2\alpha_1+2\alpha_2+2\alpha_3)}\right.$$

$$+\frac{s_1s_2\alpha_3(2\alpha_1+\alpha_2+\alpha_3)}{\{1-s_1(2\alpha_1+\alpha_2+\alpha_3)\}\{1-s_2(2\alpha_1+2\alpha_2+\alpha_3)\}}$$

$$+\frac{2s_1s_3\alpha_2(\alpha_1+\alpha_2+\alpha_3)}{\{1-s_1(2\alpha_1+\alpha_2+\alpha_3)\}\{1-s_3(2\alpha_1+2\alpha_2+2\alpha_3)\}}$$

$$\left.+\frac{2s_2s_3\alpha_1(2\alpha_1+2\alpha_2+\alpha_3)}{\{1-s_2(2\alpha_1+2\alpha_2+\alpha_3)\}\{1-s_3(2\alpha_1+2\alpha_2+2\alpha_3)\}}\right].$$

57. The general algebraical result, interpreted arithmetically, shows that

$$\tfrac{1}{2}\frac{1}{\{1-s_1(2\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(2\alpha_1+2\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)\}}$$

is a generating function for the enumeration of the compositions of multipartite numbers.

58. The original generating function of the earlier sections has by the addition of ineffective terms become factorized, and can thus be dissected for detailed examination.

The process seems to be analogous to the chemical operation by which the addition of a flux causes an element to be the more easily melted.

As a direct consequence of the geometrical representation of the compositions of multipartite numbers on a reticulation it is of great interest.

59. To resume. The number of compositions of the multipartite number $\overline{p_1p_2\ldots p_n}$ is the coefficient of

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

in the expanded product

$$\tfrac{1}{2}(2\alpha_1+\alpha_2+\ldots+\alpha_n)^{p_1}(2\alpha_1+2\alpha_2+\ldots+\alpha_n)^{p_2}\ldots(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)^{p_n},$$

or we may say that it is the coefficient of the symmetric function

$$\Sigma\,\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

in the development of the symmetric function

$$\tfrac{1}{2}\Sigma(2\alpha_1+\alpha_2+\ldots+\alpha_n)^{p_1}(2\alpha_1+2\alpha_2+\ldots+\alpha_n)^{p_2}\ldots(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)^{p_n}$$

in a sequence of monomial symmetric functions.

60. The unsymmetrical form gives a direct connection between the numbers of the compositions and of the permutations of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}.$$

In the unsymmetrical product the term $\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$, attached to some numerical coefficient, arises in connection with every permutation of the letters in the term. One factor of such coefficient is manifestly

$$2^{p_1-1};$$

if the letter α_2, in a particular permutation, occur q_2 times in the last $p_2 + p_3 + \ldots + p_n$ places of the permutation there will be a factor

$$2^{q_2};$$

further, if the letter α_s, in the same permutation, occur q_s times in the last $p_s + p_{s+1} + \ldots + p_n$ places of the permutation there will be a factor

$$2^{q_s}.$$

Hence for the permutation considered there arises a term

$$2^{p_1-1+q_2+q_3+\ldots+q_n},$$

and the number of compositions must therefore be

$$2^{p_1-1}\Sigma\, 2^{q_2+q_3+\ldots+q_n},$$

the summation being taken in regard to every permutation of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}.$$

E.g., to find in this way the number of compositions of the bipartite $\overline{22}$, we have the following scheme :—

			q_2
$p_1 = 2$	$\alpha_1\alpha_1$	$\alpha_2\alpha_2$	2
	$\alpha_1\alpha_2$	$\alpha_1\alpha_2$	1
	$\alpha_1\alpha_2$	$\alpha_2\alpha_1$	1
	$\alpha_2\alpha_1$	$\alpha_1\alpha_2$	1
	$\alpha_2\alpha_1$	$\alpha_2\alpha_1$	1
	$\alpha_2\alpha_2$	$\alpha_1\alpha_1$	0

Therefore

$$F(22) = 2^{2-1}(2^2 + 2^1 + 2^1 + 2^1 + 2^1 + 2^0) = 26.$$

61. The unsymmetrical product may be written in the form

$$2^{p_1+p_2+\dots+p_n-1}\{\alpha_1+\tfrac{1}{2}(\alpha_2+\dots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\tfrac{1}{2}(\alpha_3+\dots+\alpha_n)\}^{p_2}\dots$$
$$\dots\{\alpha_1+\alpha_2+\dots+\alpha_n\}^{p_n}$$

and herein the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$ may be written

$$2^{\Sigma p-1}\overset{s}{\Sigma}\,C_s(\tfrac{1}{2})^s=\overset{s}{\Sigma}C_s2^{\Sigma p-s-1},$$

where C_0, C_1, C_2, . . . are certain integers.

Observe that C_s is the coefficient of $\lambda^s\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$ in the product

$$\{\alpha_1+\lambda(\alpha_2+\dots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\lambda(\alpha_3+\dots+\alpha_n)\}^{p_2}\dots\{\alpha_1+\alpha_2+\dots+\alpha_n\}^{p_n}$$

Considering the reticulation of the multipartite number $\overline{p_1p_2\dots p_n}$, suppose there to be D_s lines of route which possess exactly s essential nodes. Upon these lines of route are represented

$$D_s2^{\Sigma p-s-1}$$

distinct compositions, and the whole number of compositions is manifestly

$$\Sigma\,D_s2^{\Sigma p-s-1}.$$

Hence

$$\Sigma\,D_s2^{\Sigma p-s-1}=\Sigma C_s2^{\Sigma p-s-1},$$

or $D_s=C_s$ presumably (see *post*, Sec. 5, Art. 71), that is to say, the number of distinct lines of route through the reticulation which possess exactly s essential nodes is the coefficient of

$$\lambda^s\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$$

in the product

$$\{\alpha_1+\lambda(\alpha_2+\dots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\lambda(\alpha_3+\dots+\alpha_n)\}^{p_2}\dots\{\alpha_1+\alpha_2+\dots+\alpha_n\}^{p_n}.$$

62. To interpret the product arithmetically we find that for a particular permutation of the letters in

$$\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$$

the factor λ^s will occur if

α_2	occurring	r_2	times in the first	p_1	places of the	permutation
α_3	,,	r_3	,,	p_1+p_2	,,	,,
. . . .	. . . .	. . . .	. . . .	. . . .	. . . .	. . . .
α_n	,,	r_n	,,	$p_1+p_2+\dots+p_{n-1}$	,,	,,

the sum $r_2+r_3+\dots+r_n=s$.

63. Hence the number of lines of route through the reticulation which possess exactly s essential nodes is equal to the number of permutations of the letters in

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n},$$

for which

$$r_2+r_3+\ldots+r_n=s;$$

r_t denoting the number of times that the letter α_t occurs in the first

$$p_1+p_2+\ldots+p_{t-1}$$

places of permutation.

64. We are at once led to an important identity of enumeration in the pure theory of permutations. Following the nomenclature of a previous section of the investigation, a line of route through the reticulation is traced out by a succession of steps, each step being an α_1 step or an α_2, &c., or an α_n step.

The whole length of the line of route there are altogether p_1, α_1 steps, p_2, α_2 steps, &c., . . . p_n, α_n steps.

Without regard to the characteristic of lines of route in respect of essential nodes, the whole number of lines of route is equal to the number of different orders in which these steps can be taken, viz., equal to the whole number of permutations of the letters in

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}.$$

An essential node occurs whenever a step α_u *immediately* precedes a step α_t, where $u>t$.

In the corresponding permutation there is a contact

$$\alpha_u\alpha_t,\quad u>t,$$

which may be called a *major* contact.

Hence a line of route with s essential nodes is represented by a permutation with s major contacts. We have thus the theorem :—

"In the reticulation of the multipartite number $\overline{p_1p_2\ldots p_n}$ the number of lines of route which possess exactly s essential nodes is equal to the number of permutations of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n},$$

which possess exactly s major contacts."

"Calling a contact $\alpha_u\alpha_t$ a major contact when $u>t$ the number of permutations of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n},$$

which possess exactly s major contacts, is given by the coefficient of

$$\lambda^s \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$$

in the product

$$\{\alpha_1 + \lambda(\alpha_2 + \ldots + \alpha_n)\}^{p_1} \{\alpha_1 + \alpha_2 + \lambda(\alpha_3 + \ldots + \alpha_n)\}^{p_2} \ldots \{\alpha_1 + \alpha_2 + \ldots + \alpha_n\}^{p_n}."$$

I am not aware if this problem has been previously solved in this form, or, indeed, if it has ever been attacked before.*

65. "The number of permutations of the letters in the product

$$\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n},$$

which possess exactly s major contacts, is equal to the number of permutations for which

$$r_2 + r_3 + \ldots + r_n = s,$$

r_t denoting the number of times that the latter α_t occurs in the first

$$p_1 + p_2 + \ldots + p_{t-1}$$

places of the permutation."

66. It is easy to obtain a refinement of these theorems which has been foreshadowed by the detailed bipartite and tripartite cases which have preceded. Major contacts in a permutation, and, consequently, also the essential nodes along a line of route, are of $\binom{n}{2}$ different kinds.

Consider the product

$$(\alpha_1 + \lambda_{21}\alpha_2 + \lambda_{31}\alpha_3 + \ldots + \lambda_{n1}\alpha_n)^{p_1} (\alpha_1 + \alpha_2 + \lambda_{32}\alpha_3 + \lambda_{42}\alpha_4 + \ldots + \lambda_{n2}\alpha_n)^{p_2} \ldots (\alpha_1 + \alpha_2 + \ldots + \lambda_{n,n-1}\alpha_n)^{p_{n-1}} (\alpha_1 + \alpha_2 + \ldots + \alpha_n)^{p_n}.$$

The coefficient of $\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$ consists of a number of monomial products of the quantities λ, each attached to a numerical coefficient. Of these products a certain number are of a definite degree s. The sum of the coefficients of these products gives the number of permutations of the letters in

$$\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$$

which possess exactly s major contacts.

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* See a paper by H. FORTY, M.A., "On Contact and Isolation, a Problem in Permutations," 'Proceedings London Mathematical Society,' vol. 15.

Let one such product with its attached numerical coefficient be

$$\kappa\lambda_{x_1 1}{}^{\xi_1}\lambda_{x_2 1}{}^{\xi_2} \dots \lambda_{y_1 2}{}^{\eta_1}\lambda_{y_2}{}^{\eta_2} \dots \lambda_{z_1 3}{}^{\zeta_1}\lambda_{z_2 3}{}^{\zeta_2} \dots$$

The number κ enumerates a certain number of the permutations which have s major contacts, and $\Sigma\kappa$ enumerates the entire number.

The problem is to determine the particular permutations enumerated by the number κ.

If p_1 and p_2 be both greater than zero, the contact $\alpha_2\alpha_1$ must occur in one or more of the enumerated permutations.

If p_1 be zero, λ_{21} is absent, while if p_2 be zero, α_2 has no existence, and also λ_{21} does not appear.

Hence, λ_{21} only appears when both p_1 and p_2 are greater than zero. It follows that the products which involve λ_{21} only enumerate by their coefficients those permutations which possess $\alpha_2\alpha_1$ contacts. In fact, the absence of either α_1 or α_2, by causing λ_{21} to vanish, diminishes the total number of permutations which have a definite number of major contacts; this would not be the case if the terms comprising λ_{21} as a factor did not enumerate $\alpha_2\alpha_1$ contacts; if $\alpha_2\alpha_1$ contacts were not enumerated, the vanishing of α_1 or α_2 would not alter the enumeration of the permutations possessing a given number of contacts, for this given number would be complete without $\alpha_2\alpha_1$ contacts at all.

It is next to be shown that a product involving $\lambda_{21}{}^{s_{21}}$ comprises exactly s_{21}, $\alpha_2\alpha_1$ contacts.

Suppose $p_1 \nless s_{21}$, $p_2 < s_{21}$.

Then α_2 occurs less than s_{21} times.

And, therefore, $\lambda_{21}\alpha_2$ cannot be raised to the power s_{21}.

And, therefore, λ_{21} cannot occur in any term to so high a power as s_{21}.

Similarly if $p_1 < s_{21}$ it is evident that λ_{21} cannot occur to so high a power as s_{21}. Therefore, unless both p_1 and p_2 are at least as great as s_{21}, λ_{21} cannot occur in any term to so high a power as s_{21}. It follows that the products which involve $\lambda_{21}{}^{s_{21}}$ cannot enumerate permutations possessing fewer than s_{21}, $\alpha_2\alpha_1$ contacts. For suppose that the permutations enumerated possessed only σ_{21}, $\alpha_2\alpha_1$ contacts where $\sigma_{21} < s_{21}$. If p_1 or p_2 be diminished so as to be less than s_{21} the permutations possessing σ_{21}, $\alpha_2\alpha_1$ contacts would not *necessarily* be diminished in number, whilst those possessing s_{21} such contacts as well as the products involving $\lambda_{21}{}^{s_{21}}$ would certainly vanish. Hence the products involving $\lambda_{21}{}^{s_{21}}$ cannot, through their coefficients, enumerate permutations possessing fewer than s_{21}, $\alpha_2\alpha_1$ contacts. Hence the number κ enumerates permutations having at least

$$\begin{array}{lll} \xi_1, & \alpha_{x_1}\alpha_1 & \text{contacts} \\ \xi_2, & \alpha_{x_2}\alpha_1 & \quad ,, \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\ \eta_1, & \alpha_{y_1}\alpha_2 & \quad ,, \\ \eta_2, & \alpha_{y_2}\alpha_2 & \quad ,, \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ \zeta_1, & \alpha_{z_1}\alpha_3 & \quad ,, \\ \zeta_2, & \alpha_{z_2}\alpha_3 & \quad ,, \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \end{array}$$

and these numbers must represent also the exact numbers of the contacts in question because the sum

$$\Sigma\,\xi + \Sigma\,\eta + \Sigma\,\zeta + \ldots$$

gives the whole number of contacts under consideration.

The number κ obviously denotes the number of lines of route through the reticulation of the multipartite

$$\overline{p_1p_2\ldots p_n}$$

which possess

$$\begin{array}{llll} \xi_1 \text{ essential nodes of the kinds } & & & \alpha_{x_1}\alpha_1 \\ \xi_2 \qquad ,, & ,, & ,, & \alpha_{x_2}\alpha_1 \end{array}$$

and so forth.

67. By an easy arithmetical interpretation the number κ also denotes the number of permutations in which

α_{x_1} occurs ξ_1 times in the first p_1 places.
α_{x_2} ,, ξ_2 ,, ,, ,, ,,
. . . .
α_{y_1} ,, η_1 ,, between the p_1^{th} and p_2^{th} places.
α_{y_2} ,, η_2 ,, ,, ,, ,, ,,
. . . .
α_{z_1} ,, ζ_1 ,, ,, ,, p_2^{th} ,, p_3^{th} ,,
α_{z_2} ,, ζ_2 ,, ,, ,, ,, ,, ,, ,,
. . . .

§ 5. *Extension of the idea of Composition.*

68. The idea of composition is capable of enlargement from a particular point of view.

In regard to unipartite numbers, consider p units placed in a row

$$1\ 1\ 1\ 1\ 1\ 1 \ldots,$$

there are $p - 1$ spaces between them which may be occupied by algebraic symbols at pleasure. We may select a definite number of different symbols, and choose any one to occupy any one of the $p - 1$ spaces. If this definite number be k, we can, by filling each space at pleasure, arrive at k^{p-1}, different expressions involving the p units.

Clearly we may take as one of these symbols the simple unoccupied blank space,* since such space left between any two numbers, quantities, or expressions of any kinds has, in every case, a well-understood signification, not always, however, the same, in mathematical work.

If we restrict ourselves to a single symbol, only one expression involving the units is possible; choosing this symbol to be that which indicates addition we merely get

$$1 + 1 + 1 + 1 + 1 + 1 + \ldots,$$

which is p, or the number which enumerates the units. Had the chosen symbol been that denoting subtraction, the expression would have denoted $-(p-2)$; a blank space would have yielded a succession of p units; the symbol of multiplication, unity, and so forth.

All the modes of obtaining expressions from the p units may be called combinations of the first order in respect of the p units.

Passing to the case of two different symbols we may choose to employ the sign of addition and the blank space. We thus obtain 2^{p-1} different expressions which are the several compositions of the number p.

In general, expressions obtained by choosing from any k different symbols may be called combinations of order k in respect of the p fundamental units.

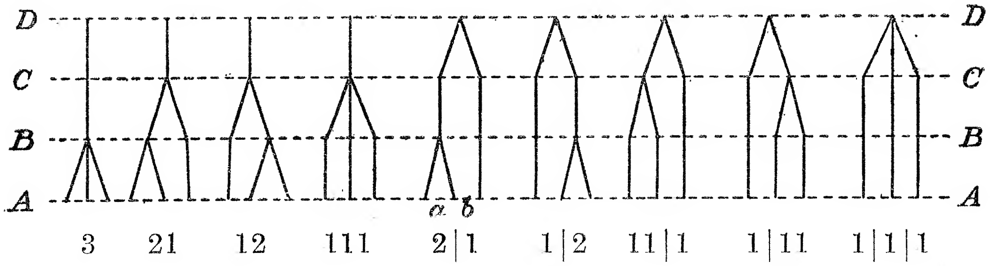
69. There is a one-to-one correspondence between these combinations and the "Trees" which have an altitude k and p terminal knots.

As an example take k and p each equal to 3, and, further, take as symbols, the sign of addition, the blank space and the symbol $|$ of unspecified signification.

The correspondence between the nine trees and the nine combinations of order three is shown below:—

---

* The unoccupied blank space might be represented by a definite symbol such as O, as suggested to me by one of the referees.



The process consists in writing down a unit for each terminal knot. The $p-1$ intervals between the units correspond to the $p-1$ inter-terminal-knot spaces. If a space leads to a bifurcation in row B the symbol is $+$; if in row C it is a blank space; if in row D it is $|$.

Thus, take the fifth tree above, we have

$$1+1|1$$

for the space a leads to a bifurcation in row B, giving the symbol $+$ and the space b leads to a bifurcation in row D giving the symbol $|$. Hence the corresponding combination

$$2|1.$$

On the same principle a tree can always be drawn to represent any combination of order k in respect of p units.

70. Passing to the case of multipartite numbers, consider tripartite numbers as representative of the general case. Arrange a row of multipartite units of the three kinds, viz. :—

$$100 \quad 100 \quad 100 \ldots 010 \quad 010 \quad 010 \ldots 001 \quad 001 \quad 001 \ldots$$

and employing two different symbols, viz., the sign of addition and the blank space, we arrive at a certain composition of the tripartite $\overline{p_1p_2p_3}$, supposing the numbers of the units 100, 010, 001 that appear in the row to be p_1, p_2, and p_3 respectively. Without altering the positions of the symbols introduced we may change the order of the units 100, 010, 001, and thus obtain other compositions. Permutations of these units between contiguous blank spaces are not permissible, so that for fixed positions of the pluses and blank spaces we do not obtain in general

$$\frac{(p_1+p_2+p_3)!}{p_1!\,p_2!\,p_3!}$$

compositions, but some lesser number.

This arises from the commutative nature of the symbol $+$.

Had neither of the symbols employed been such as obey the commutative law of algebra, the whole number of combinations of the second order would have been simply

$$2^{p_1+p_2+p_3-1}\frac{(p_1+p_2+p_3)!}{p_1!\,p_2!\,p_3!}.$$

Thus, in this case, the symbol $+$ introduces a complexity into the theory of compositions.

Choosing similarly from k different symbols, none of which are commutative, there are

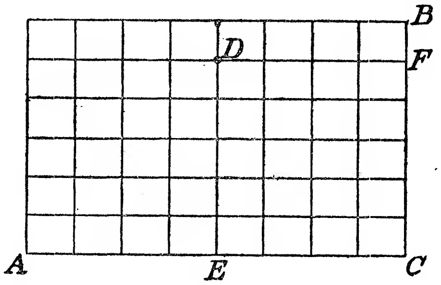
$$k^{p_1+p_2+p_3-1}\frac{(p_1+p_2+p_3)!}{p_1!\,p_2!\,p_3!},$$

combinations of order k, and the general generating function may clearly be written

$$\frac{1}{k}\cdot\frac{1}{1-k(\alpha_1+\alpha_2+\ldots+\alpha_n)};$$

but there is a lesser number of combinations if one or more of the symbols be commutative.

71. For the purpose of this investigation the most interesting case to consider is that in which one symbol, viz., the blank space, is non-commutative, and the remaining $k-1$ symbols commutative. This species of combination is not brought under view merely for the purpose of adding and discussing new complexity. Its introduction is absolutely vital to the investigation as clinching and confirming a conclusion which was momentarily assumed during the consideration of the compositions of multipartite numbers. (See *ante*, Art. 61.) Consider the reticulation of a multipartite number. That of a bipartite number will suffice as indicative of the general case.



A combination of order k (of the nature under view) is regarded as having *m parts* when $m-1$ blank space symbols occur in the combination.

Let us first enquire how many combinations possess only a single part. We require nodes of k different kinds, viz., a blank space node and $k-1$ other different nodes. A blank space node may be either essential or non-essential, but an essential node must be also a blank space node. The only line of route through the reticulation which does not involve an essential (that is a blank space) node is ACB. Consequently the graphs of all the combinations having but one part must be along the line of route ACB. Similarly in the reticulation of the multipartite number

$$\overline{p_1 p_2 \ldots p_n},$$

the graphs of all the one-part combinations will be along the line of route traced out by p_1, α_1 segments, p_2, α_2 segments, and so on in succession. The $k-1$ different symbols at disposal may be placed at pleasure at $p_1 + p_2 + \ldots + p_n - 1$ points along this line of route. Hence

$$(k-1)^{p_1 + p_2 + \ldots + p_n - 1}$$

different one-part combinations are obtainable.

In other words the generating function for such combinations is

$$h_1 + (k-1)\,h_2 + (k-1)^2 h_3 + \ldots + (k-1)^{n'-1} h_{n'} \quad (n' = \infty,$$

where in a previous notation h_n is the homogeneous product sum, degree n, of the quantities

$$\alpha_1, \alpha_2, \ldots \alpha_n.$$

72. Next as to the combinations which have two parts. At any point D of the reticulation place a blank space node. All two-part combinations whose graphs pass through D must follow the line of route AEDFB, for otherwise an additional essential (and blank space) node would be introduced. The whole combination may be split up into a one-part combination along the line of route AED, followed first by a blank space, and then by a one-part combination along the line of route DFB. All the two part combinations whose graphs pass through the point D are obtained by associating every one-part combination in the reticulation AD with every one-part combination in the reticulation DB.

Hence the whole number of combinations, having two parts, of the multipartite number

$$\overline{p_1 p_2 \ldots p_n}$$

is

$$\Sigma\,(k-1)^{p'_1 + p'_2 + \ldots + p'_n - 1} (k-1)^{p''_1 + p''_2 + \ldots + p''_n - 1}$$

where

$$p'_1 + p''_1 = p_1; \;\; p'_2 + p''_2 = p_2, \ldots p'_n + p''_n = p_n.$$

Hence, the generating function for two-part combinations is

$$\{h_1 + (k-1)\,h_2 + (k-1)^2 h_3 + \ldots + (k-1)^{n'-1} h_{n'}\}^2. \qquad n' = \infty.$$

73. Pursuing this chain of reasoning it is completely manifest that the generating function of m-part combinations is

$$\{h_1 + (k-1)\,h_2 + (k-1)^2\,h_3 + \ldots + (k-1)^{n'-1}h_{n'}\}^m \qquad (n' = \infty\,,$$

and hence the complete generating function of combinations of the multipartite number

$$\overline{p_1p_2 \cdot \ \cdot p_n}$$

is

$$\frac{h_1 + (k-1)\,h_2 + \ldots + (k-1)^{n'-1}h_{n'}}{1 - h_1 - (k-1)\,h_2 - \ldots - (k-1)^{n'-1}h_{n'}} \qquad (n' = \infty\,,$$

which is effectively the same as

$$\frac{1}{k}\cdot\frac{1}{1 - k\,\Sigma\,\alpha_1 + k\,(k-1)\,\Sigma\,\alpha_1\alpha_2 - k\,(k-1)^2\,\Sigma\,\alpha_1\alpha_2\alpha_3 + \ldots + (-)^n k\,(k-1)^{n-1}\alpha_1\alpha_2\ldots\alpha_n}\,;$$

for the latter is obtained by adding the fraction $1/k$ to the former and then transforming from homogeneous product sums to elementary symmetric functions.

Just as in the case of $k = 2$, corresponding to compositions, this generating function admits of an important transformation to a factorized redundant form.

74. In the above fraction put $s_1\alpha_1, s_2\alpha_2, \ldots s_n\alpha_n$ for $\alpha_1, \alpha_2, \ldots \alpha_n$ respectively; it may then be written

$$\frac{k-1}{k}\cdot\frac{1}{-1 + k\,\{1-(k-1)\,s_1\alpha_1\}\,\{1-(k-1)\,s_2\alpha_2\}\ldots\{1-(k-1)\,s_n\alpha_n\}} = \frac{1}{k\mathrm{N}}\,.$$

For brevity put

$$\mathrm{S}_t = s_t\,(k\alpha_1 + k\alpha_2 + \ldots + k\alpha_t + \alpha_{t+1} + \ldots + \alpha_n) = s_t\,(\mathrm{A}_t + k\alpha_t)$$
$$b_1 = \Sigma\,s_1\alpha_1, \qquad b_2 = \Sigma\,s_1s_2\alpha_1\alpha_2, \qquad \text{\&c.,}$$
$$\mathrm{M} = (1 - ks_1\alpha_1)\,(1 - ks_2\alpha_2)\ldots(1 - ks_n\alpha_n),$$

and

$$\frac{1}{k}\cdot\frac{1}{(1-\mathrm{S}_1)\,(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)} = \frac{1}{k\mathrm{D}}\,.$$

It will be shown that $\frac{1}{k\mathrm{D}}$ is a generating function equivalent to the former in regard to terms which are products of powers of

$$s_1\alpha_1,\ s_2\alpha_2,\ \ldots\ s_n\alpha_n\,;$$

for

$$\mathrm{M}-\mathrm{N}=\{k^2-k(k-1)\}b_2-\{k^3-k(k-1)^2\}b_3+\ldots+(-)^n\{k^n-k(k-1)^{n-1}\}b_n,$$

and thence

$$\frac{\mathrm{N}}{\mathrm{D}}=\frac{\mathrm{M}}{\mathrm{D}}-\frac{(\mathrm{M}-\mathrm{N})}{\mathrm{D}}=\prod_1^n\left(1+\frac{s_t\mathrm{A}_t}{1-\mathrm{S}_t}\right)-\frac{\{k^2-k(k-1)\}b_2-\ldots+(-)^n\{k^n-k(k-1)^{n-1}\}b_n}{(1-\mathrm{S}_1)(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)}.$$

75. The investigation proceeds precisely as in the case of $k = 2$ with the following result. The fraction

$$\frac{1}{k}\cdot\frac{1}{\{1-s_1(k\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(k\alpha_1+k\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(k\alpha_1+k\alpha_2+\ldots+k\alpha_n)\}}$$

is equal to the product of the fraction

$$\frac{1}{k}\cdot\frac{1}{1-k\Sigma s_1\alpha_1+k(k-1)\Sigma s_1s_2\alpha_1\alpha_2-\ldots+(-)^nk(k-1)^{n-1}s_1s_2\ldots s_n\alpha_1\alpha_2\ldots\alpha_n},$$

and the series

$$1+\Sigma\frac{k(\mathrm{A}_{t_1}+\alpha_{t_1})(\mathrm{A}_{t_2}+\alpha_{t_2})\ldots(\mathrm{A}_{t_u}+\alpha_{t_u})-(\mathrm{A}_{t_1}+k\alpha_{t_1})(\mathrm{A}_{t_2}+k\alpha_{t_2})\ldots(\mathrm{A}_{t_u}+k\alpha_{t_u})}{(k-1)(1-\mathrm{S}_{t_1})(1-\mathrm{S}_{t_2})\ldots(1-\mathrm{S}_{t_u})},$$

the summation having regard to every selection of u integers from the series $1, 2, 3, \ldots n$, and u takes all values from 1 to $n-1$.

As in the former case the relation

$$\mathrm{A}_t+k\alpha_t=\mathrm{A}_{t+1}+\alpha_{t+1},$$

which becomes

$$\mathrm{A}_{t_u}+k\alpha_{t_u}=k(\mathrm{A}_{t_1}+\alpha_{t_1})$$

where t_u and t_1 are the highest and lowest suffixes present, shows that the terms under the summation sign do not involve any products of $s_1\alpha_1, s_2\alpha_2, \ldots s_n\alpha_n$ only, and therefore as far as concerns the generating function may be put equal to zero.

76. Hence the number of combinations of order k of the multipartite number

$$\overline{p_1p_2\ldots p_n}$$

is the coefficient of $(s_1\alpha_1)^{p_1}(s_2\alpha_2)^{p_2}\ldots(s_n\alpha_n)^{p_n}$ in the expansion of the generating function

$$\frac{1}{k}\cdot\frac{1}{\{1-s_1(k\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(k\alpha_1+k\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(k\alpha_1+\ldots+k\alpha_n)\}}$$

that is to say it is the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2} \dots \alpha_n^{p_n}$ in the product

$$\frac{1}{k}(k\alpha_1 + \alpha_2 + \dots + \alpha_n)^{p_1}(k\alpha_1 + k\alpha_2 + \alpha_3 + \dots + \alpha_n)^{p_2} \dots (k\alpha_1 + k\alpha_2 + \dots + k\alpha_n)^{p_n},$$

which may be written

$$k^{p_1+p_2+\dots+p_n-1}\{\alpha_1 + \frac{1}{k}(\alpha_2 + \dots + \alpha_n)\}^{p_1}\{\alpha_1 + \alpha_2 + \frac{1}{k}(\alpha_3 + \dots + \alpha_n)\}^{p_2}$$
$$\dots \{\alpha_1 + \alpha_2 + \dots + \alpha_n\}^{p_n}.$$

The coefficient of $\alpha_1^{p_1}\alpha_2^{p_2} \dots \alpha_n^{p_n}$ is

$$\overset{s}{\Sigma}\, C_s\, k^{\Sigma p - s - 1},$$

where C_s is the coefficient of

$$\lambda^s\, \alpha_1^{p_1}\alpha_2^{p_2} \dots \alpha_n^{p_n}$$

in the product

$$\{\alpha_1 + \lambda(\alpha_2 + \dots + \alpha_n)\}^{p_1}\{\alpha_1 + \alpha_2 + \lambda(\alpha_3 + \dots + \alpha_n)\}^{p_2} \dots \{\alpha_1 + \alpha_2 + \dots + \alpha_n\}^{p_n}.$$

If in the reticulation of the multipartite number there be D_s lines of route which possess exactly s essential nodes,

$$D_s\, k^{\Sigma p - s - 1}$$

combinations of order k may be represented upon these lines. Hence the whole number of combinations is

$$\overset{s}{\Sigma}\, D_s\, k^{\Sigma p - s - 1}.$$

Hence

$$\overset{s}{\Sigma}\, D_s\, k^{\Sigma p - s - 1} = \overset{s}{\Sigma}\, C_s\, k^{\Sigma p - s - 1}$$

a relation which is true for *all positive integral values of k.*

77. Hence

$$D_s = C_s$$

the important relation temporarily assumed in the investigation concerning compositions. (Art. 61.)

78. The theorem thus established, viz., that the number of distinct lines of route through the reticulation of the multipartite

$$\overline{p_1 p_2 \dots p_n},$$

which possess exactly s essential nodes, is given by the coefficient of

$$\lambda^s \alpha_1^{p_1}\alpha_2^{p_2} \dots \alpha_n^{p_n}$$

in the product

$$\{\alpha_1 + \lambda(\alpha_2 + \ldots + \alpha_n)\}^{p_1} \{\alpha_1 + \alpha_2 + \lambda(\alpha_3 + \ldots + \alpha_n)\}^{p_2} \ldots \{\alpha_1 + \alpha_2 + \ldots + \alpha_n\}^{p_n}$$

gives the theorem in compositions :—

"The number of compositions of the multipartite

$$\overline{p_1 p_2 \ldots p_n}$$

which possess exactly s contacts of

$$n - u \text{ zeros with } \quad - 1 \text{ zeros,}$$

subject to the condition $u > t$, is

$$C_s 2^{p_1 + p_2 + \ldots + p_n - s - 1}$$

where C_s is the coefficient of

$$\lambda^s \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$$

in the above-mentioned product."

79. Also the further theorem :—

"The number of compositions of the multipartite

$$\overline{p_1 p_2 \ldots p_n}$$

which possess exactly

$s_{u_1 t_1}$ contacts of $n - u_1$ zeros with $t_1 - 1$ zeros.
$s_{u_2 t_2}$ „ $n - u_2$ „ $t_2 - 1$ „
. . . .

is the product of

$$2^{p_1 + p_2 + \ldots + p_n - s - 1} \qquad [s = \Sigma s_{ut}]$$

and the coefficient of

$$\lambda_{u_1 t_1}^{s_{u_1 t_1}} \lambda_{u_2 t_2}^{s_{u_2 t_2}} \ldots \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$$

in the product

$$(\alpha_1 + \lambda_{21}\alpha_2 + \ldots + \lambda_{n1}\alpha_n)^{p_1} (\alpha_1 + \alpha_2 + \lambda_{32}\alpha_3 + \ldots + \lambda_{n2}\alpha_n)^{p_2} \ldots (\alpha_1 + \alpha_2 + \ldots + \alpha_n)^{p_n}.$$

80. The generating function which enumerates the combinations of order k of multipartite numbers may be written

$$\frac{1}{k} + \frac{H'}{k-1} + \frac{H'^2}{(k-1)^2} + \frac{H'^3}{(k-1)^3} + \ldots + \frac{H'^m}{(k-1)^m} + \ldots$$

where

$$\alpha'_1 = (k-1)\,\alpha_1, \qquad \alpha'_2 = (k-1)\,\alpha_2, \text{ \&c.}$$

h'_m is the homogeneous product sum of degree m of the quantities

$$\alpha'_1, \alpha'_2, \ldots \alpha'_n,$$

and

$$\mathrm{H}' = h'_1 + h'_2 + h'_3 + \ldots$$

The coefficient of

$$\alpha_1^{p_1}\alpha_2^{p_2} \ldots \alpha_n^{p_n}$$

in

$$\frac{\mathrm{H}'^m}{(k-1)^m}$$

enumerates, in respect of the multipartite $\overline{p_1 p_2 \ldots p_n}$, the number of combinations having m parts.

From previous work this coefficient is

$$(k-1)^{p_1+p_2+\ldots+p_n-m} f(p_1 p_2 \ldots p_n, m),$$

where

$$\begin{aligned} f(p_1 p_2 \ldots p_n, m) = &\binom{p_1+m-1}{p_1}\binom{p_2+m-1}{p_2}\ldots\binom{p_n+m-1}{p_n} \\ &- \binom{m}{1}\binom{p_1+m-2}{p_1}\binom{p_2+m-2}{p_2}\ldots\binom{p_2+m-2}{p_n} \\ &+ \&c., \end{aligned}$$

to m terms.

81. Hence the whole number of combinations is

$$(k-1)^{\Sigma p-1} f(p_1 p_2 \ldots p_n, 1) + (k-1)^{\Sigma p-2} f(p_1 p_2 \ldots p_n, 2) + .$$

a result which is immediately obtainable from the reticulation.

82. There exists a very interesting correspondence between the compositions of the multipartite

$$\overline{1^{n-1}}$$

into k parts, zeros not excluded, and the combinations of order k of the unipartite number

$$n,$$

zeros excluded.

The generating function for the compositions of multipartite numbers into k parts zeros not excluded, is

$$(1 + h_1 + h_2 + \ldots)^k = (1-\alpha_1)^{-k}(1-\alpha_2)^{-k} \ldots (1-\alpha_n)^{-k},$$

hence the number in the case of $\overline{p_1 p_2 \ldots p_n}$ is

$$\binom{k+p_1-1}{p_1}\binom{k+p_2-1}{p_2}\binom{k+p_3-1}{p_3}\ldots\binom{k+p_n-1}{p_n},$$

which for the multipartite $\overline{1^{n-1}}$ is

$$k^{n-1}.$$

This expression also gives (*ante*) the number of combinations of order k of the unipartite number

$$n,$$

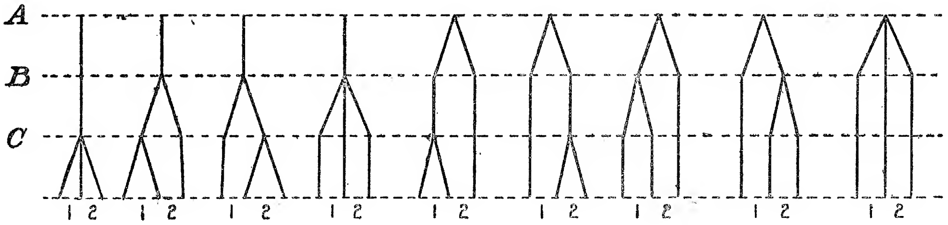
zeros excluded.

83. The correspondence may be shown by reference to the "Theory of Trees."

The trees to be considered are of altitude k and have n terminal knots.

For simplicity take $n = k = 3$.

The trees are



which, as shown above, represent combinations of order 3 of the unipartite number 3. These are

$$3 \quad 21 \quad 12 \quad 111 \quad 2|1 \quad 1|2 \quad 11|1 \quad 1|11 \quad 1|1|1,$$

the number of parts (observe that blank space symbols are between adjacent parts) being equal to the number of bifurcations in the second row from the top, increased by unity.

Now these trees may be interpreted so as to represent the compositions of the multipartite $(\overline{1^2})$ into three parts, zero parts not excluded, by attending to the connection between the two bifurcations of each tree and the two inter-terminal-knot spaces.

In any row of knots in a tree we have or we have not bifurcations, and each bifurcation communicates either with the first or with the second inter-terminal-knot space.

Beginning with the row marked A, if we find no bifurcation we write $\overline{00}$; if we find a bifurcation communicating with the first space but not with the second we write $\overline{10}$; if with the second space and not with the first $\overline{01}$; if there be two bifurcations, which necessarily in the present case communicate with both spaces, we write $\overline{11}$; and proceeding in the same way with the rows B and C in succession we will have finally written down three bipartite parts constituting a composition of the multipartite $\overline{11}$.

We thus obtain, beginning with the left hand tree,

$$(\overline{00}\ \overline{00}\ \overline{11}),\ (\overline{00}\ \overline{01}\ \overline{10}),\ (\overline{00}\ \overline{10}\ \overline{01}),\ (\overline{00}\ \overline{11}\ \overline{00}),\ (\overline{01}\ \overline{00}\ \overline{10}),\ (\overline{10}\ \overline{00}\ \overline{01}),$$
$$(\overline{01}\ \overline{10}\ \overline{00}),\ (\overline{10}\ \overline{01}\ \overline{00}),\ (\overline{11}\ \overline{00}\ \overline{00}),$$

in order of correspondence with the above written combinations, of order 3, of the unipartite number 3.

84. The process is perfectly general. Every tree of altitude k is representative alike of a combination of order k of a unipartite number, zeros excluded, and of a composition of a unitary multipartite, zeros not excluded, and each combination or composition of the nature considered is uniquely represented by a tree.

The number of compositions of the multipartite $\overline{1^{n-1}}$, zeros not excluded, into k or fewer parts is

$$1^{n-1} + 2^{n-1} + \ldots + k^{n-1}$$

a number which also represents the number of the aggregate of the combinations of the number n, of orders 1, 2, . . . k, zero parts not excluded.

The interesting fact here brought to light is the connection between the unipartite numbers and the unitary multipartite numbers.

85. We have seen that k^{m-1} expresses the number of combinations of order k possessed by the unipartite number m. Each combination involves a certain number of the k different symbols and we may inquire the number of combinations which involve exactly p out of the k symbols. It is clear that one combination can be formed which involves any one symbol and none of the others; hence, k combinations involve but a single symbol. Two out of k symbols may be selected in $\binom{k}{2}$ different ways; for each such selection we must take the number of combinations of the number m, of order 2, and subtract the number of these in which but a single symbol appears. Hence, the number of combinations involving exactly two symbols is

$$\binom{k}{2}(2^{m-1} - 2).$$

Similarly three out of k symbols may be selected in $\binom{k}{3}$ ways, and for each selection we take the whole number of combinations of order 3 and subtract those of them which involve exactly two symbols or exactly one.

Hence we arrive at the number

$$\binom{k}{3}\{3^{m-1} - 3\,(2^{m-1} - 2) - 3\}$$
$$= \binom{k}{3}\{3^{m-1} - 3\,.\,2^{m-1} + 3\,.\,1^{m-1}\}.$$

In this way it is easy to see that the number of combinations involving exactly p symbols is

$$\binom{k}{p}\{p^{m-1} - p\,(p-1)^{m-1} + \binom{p}{2}(p-2)^{m-1} - \ldots + (-)^{p+1}p\}$$

$$= \binom{k}{\ }\,\Delta^p\,(0^{m-1})$$

in the notation of finite differences.

We have now the well known identity

$$k^{m-1} = \binom{k}{1}\Delta\,(0^{m-1}) + \binom{k}{2}\Delta^2\,(0^{m-1}) + \ldots + \binom{k}{p}\Delta^p\,(0^{m-1}) + \ldots + \binom{k}{k}\Delta^k\,(0^{m-1}),$$

and we have seen its interpretation in the theory of the combinations of order k of a given unipartite number m.

TABLES of Compositions of Multipartite Numbers.

No.	1	part	Total
1	1	. . .	1

No.	1	2	parts	Total
2	1	1	. . .	2
$\overline{1^2}$	1	2	. . .	3

No.	1	2	3	parts	Total
3	1	2	1	. . .	4
$\overline{21}$	1	4	3	. . .	8
$\overline{1^3}$	1	6	6	. . .	13

No.	1	2	3	4	parts	Total
4	1	3	3	1	. . .	8
$\overline{31}$	1	6	9	4	. . .	20
$\overline{2^2}$	1	7	12	6	. . .	26
$\overline{21^2}$	1	10	21	12	. . .	44
$\overline{1^4}$	1	14	36	24	. . .	75

No.	1	2	3	4	5	parts	Total
5	1	4	6	4	1	. .	16
$\overline{41}$	1	8	18	16	5	. .	48
$\overline{32}$	1	10	27	28	10	. .	76
$\overline{31^2}$	1	14	45	52	20	. .	132
$\overline{2^21}$	1	16	57	72	30	. .	176
$\overline{21^3}$	1	22	93	132	60	. .	308
$\overline{1^4}$	1	30	150	240	120	. .	541

No.	1	2	3	4	5	6	parts	Total
6	1	5	10	10	5	1	. .	32
$\overline{51}$	1	10	30	40	25	6	. .	112
$\overline{42}$	1	13	48	76	55	15	. .	208
$\overline{41^2}$	1	18	78	136	105	30	. .	368
$\overline{3^2}$	1	14	55	92	70	20	. .	252
$\overline{321}$	1	22	111	220	190	60	. .	604
$\overline{31^3}$	1	30	177	388	360	120	. .	1076
$\overline{2^3}$	1	25	138	294	270	90	. .	818
$\overline{2^21^2}$	1	34	219	516	510	180	. .	1460
$\overline{21^4}$	1	46	345	900	960	360	. .	2612
$\overline{1^6}$	1	62	540	1560	1800	720	. .	4683

No.	1	2	3	4	5	6	7	parts	Total
7	1	6	15	20	15	6	1	. .	64
$\overline{61}$	1	12	45	80	75	36	7	. .	256
$\overline{52}$	1	16	75	160	175	96	21	. .	544
$\overline{51^2}$	1	22	120	280	325	186	42	. .	976
$\overline{43}$	1	18	93	216	255	150	35	. .	768
$\overline{421}$	1	28	183	496	655	420	105	. .	1888
$\overline{41^3}$	1	38	288	856	1205	810	210	. .	3408
$\overline{3^21}$	1	30	207	588	810	540	140	. .	2316
$\overline{32^2}$	1	34	255	772	1120	780	210	. .	3172
$\overline{321^2}$	1	46	399	1324	2050	1500	420	. .	5740
$\overline{31^4}$	1	62	621	2260	3740	2880	840	. .	10404
$\overline{2^31}$	1	52	489	1728	2820	2160	630	. .	7880
$\overline{2^21^3}$	1	70	759	2940	5130	4140	1260	. .	14300
$\overline{21^5}$	1	94	1173	4980	9300	7920	2520	. .	25988
$\overline{1^7}$	1	126	1806	8400	16800	15120	5040	. .	47293

[During the considerable time that has elapsed since this paper was read I have discovered the general theory of the transformations of Arts. 41 and 75.

Let X_1, X_2, X_3 be general linear functions of x_1, x_2, x_3 as exhibited in the Notation of the Theory of Matrices

$$(X_1, X_2, X_3) = \left(\begin{array}{ccc} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{array}\right)(x_1, x_2, x_3),$$

and consider the algebraic fraction

$$\frac{1}{(1 - s_1X_1)(1 - s_2X_2)(1 - s_3X_3)}.$$

I have established that that portion of the expansion of this fraction, which is a function of products of powers of s_1x_1, s_2x_2, s_3x_3 only, is represented by the fraction

$$\frac{1}{\mathrm{N}},$$

where

$$\begin{aligned}\mathrm{N} = 1 &- a_1s_1x_1 - b_2s_2x_2 - c_3s_3x_3 \\ &+ \mid a_1b_2 \mid s_1s_2x_1x_2 + \mid a_1c_3 \mid s_1s_3x_1x_3 + \mid b_2c_3 \mid s_2s_3x_2x_3 \\ &- \mid a_1b_2c_3 \mid s_1s_2s_3x_1x_2x_3\,;\end{aligned}$$

the notation being that in use in the Theory of Determinants.

The coefficients of N are the several co-axial minors of the determinant of the matrix defining X_1, X_2, and X_3, viz.:—

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The result is immediately deducible from the identity

$$\begin{vmatrix} \dfrac{s_1(a_1x_1 - \mathrm{X}_1)}{1 - s_1\mathrm{X}_1} - 1, & \dfrac{a_2s_1x_1}{1 - s_1\mathrm{X}_1}, & \dfrac{a_3s_1x_1}{1 - s_1\mathrm{X}_1} \\ \dfrac{b_1s_2x_2}{1 - s_2\mathrm{X}_2}, & \dfrac{s_2(b_2x_2 - \mathrm{X}_2)}{1 - s_2\mathrm{X}_2} - 1, & \dfrac{b_3s_2x_2}{1 - s_2\mathrm{X}_2} \\ \dfrac{c_1s_3x_3}{1 - s_3\mathrm{X}_3}, & \dfrac{c_2s_3x_3}{1 - s_3\mathrm{X}_3}, & \dfrac{s_3(c_3x_3 - \mathrm{X}_3)}{1 - s_3\mathrm{X}_3} - 1 \end{vmatrix}$$

$$\times \begin{vmatrix} 1 - s_1\mathrm{X}_1, & 0, & 0 \\ 0, & 1 - s_2\mathrm{X}_2, & 0 \\ 0, & 0, & 1 - s_3\mathrm{X}_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1s_1x_1 - 1, & a_2s_1x_1, & a_3s_1x_1 \\ b_1s_2x_2, & b_2s_2x_2 - 1, & b_3s_2x_2 \\ c_1s_3x_3, & c_2s_3x_3, & c_3s_3x_3 - 1 \end{vmatrix}.$$

the determinant last written being, with changed sign, the value of N.

The theorem for the case of n variables $x_1, x_2, \ldots x_n$ will be completely manifest from the above.

It appears to be one of considerable importance with regard to the generating functions which present themselves in this domain of the Theory of Numbers.

The results of its further investigation I hope to bring before the Royal Society in the near future.—Added August 25, 1893. P. A. M.]